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PREDETERMINED AND NON-PREDETERMINED VARIABLES IN RATIONAL EXPECTATIONS MODELS

Willem H. Buiter

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ABSTRACT

The distinction between predetermined and non-predetermined variables is a crucial one in rational expectations models. I consider and reject two definitions, one proposed by Blanchard and Kahn and one by Chow. Both definitions lead to possible misclassifications. Instead I propose the following definition. A variable is non-predetermined if and only if its current value is a function of current anticipations of future values of endogenous and/or exogenous variables. This definition focuses on the essential economic property of non-predetermined variables: unlike predetermined variables they can respond instantaneously to changes in expectations due to "news." The new definition also fits the structure of rational expectations models solution algorithms such as the one proposed by Blanchard and Kahn.

Willem H. Buiter
Department of Economics
London School of Economics
and Political Science
Houghton Street
London, WC2A 2AE
England

01-405-7686 ext. 406

Predetermined and non-predetermined variables in rational expectations models

The distinction between predetermined and non-predetermined variables is an extremely important one in rational expectations models. In a recent paper Blanchard and Kahn [1980] proposed the following definition. (For any variable Z_t, ${}_{t-i}Z_{t+j} \equiv E\left(Z_{t+j} | \Omega_{t-i}\right)$, where E denotes the mathematical expectation operator and Ω_t the information set at t conditioning expectations formed at t). "A predetermined variable [X_{t+1}] is a function only of variables known at time t, that is of variables in Ω_t, so that $X_{t+1} = {}_tX_{t+1}$ whatever the realization of the variables in Ω_{t+1}. A non-predetermined variable P_{t+1} can be a function of any variable in Ω_{t+1}, so that we can conclude that $P_{t+1} = {}_tP_{t+1}$ only if the realizations of all variables in Ω_{t+1} are equal to their expectations conditional on Ω_t." (Blanchard and Kahn [1980, p. 1305]).

For concreteness, consider the linear first order system of Blanchard and Kahn:

$$(1) \quad \begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} Z_t$$

$$X_{t_o} = X_o$$

$\begin{bmatrix} X_t \\ P_t \end{bmatrix}$ is the state vector, Z_t the vector of exogenous or forcing variables.

X_t is an n-vector, P_t an m-vector and Z_t a k-vector.

$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ represents a partitioning of A conformable with X and P. γ_1 is the first n rows of γ and γ_2 the last m rows.

If A is of full rank, $n + m$ linearly independent boundary conditions for the state variables are required to obtain a unique solution. For the predetermined variables, X, these boundary conditions take the form of n initial values at $t = t_o$, say, determined by the past history of the system. At a point in time, t, the predetermined variables X_t cannot respond to changes in expectations formed at t, due to new information becoming available at t, about the future behaviour of the endogenous or exogenous variables.

For the m non-predetermined variables, the boundary conditions do not take the form of historically given initial values. Instead they typically take the form of the transversality or terminal condition that the homogeneous part of the equation system,

$$(2) \quad \begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix}$$

converges to zero as $t \to \infty$ from any X_o.

If the characteristic equation of A has n stable (modulus < 1) roots and m unstable (modulus > 1) roots, this convergence condition provides the "missing" m independent boundary conditions required for a unique solution (Blanchard and Kahn [1980]). P_t represents in economic applications such "forward-looking" variables as asset prices determined in efficient auction markets dominated by arbitrageurs and speculators endowed with forward-looking rational expectations. X_t represents such "backward-looking" variables as the physical capital stock and, in some Keynesian models, temporarily fixed money wages or prices.

It seems self-evident that Blanchard and Kahn's definition of a predetermined variable is far too restrictive to capture the economic meaning of the concept. It is also unnecessarily restrictive for the validity of the solution method for (1) that they present. Consider e.g. the naive scalar autoregressive model in (3). ε is a white noise disturbance term.

$$(3) \quad y_{t+1} = \alpha y_t + \varepsilon_{t+1}$$

Clearly $y_{t+1} = \alpha y_t + \varepsilon_{t+1} \not\Leftrightarrow E\left(y_{t+1} | \Omega_t\right) = \alpha y_t$. y_{t+1} doesn't have to be perfectly predictable on the basis of information available at time t , for it to be a predetermined variable in the economic sense or for Blanchard and Kahn's solution method to be applicable. Equation (3) is the special case of (1) with $A_{11} = \alpha$, $A_{12} = A_{21} = A_{22} = 0$ $\gamma_1 = 1$, $\gamma_2 = 0$ and $Z_t = \varepsilon_{t+1}$. Its "final form" solution, given the initial condition $y_{t-t_o} = y_o$ is

$$(4) \quad y_t = \alpha^{t_o} y_o + \sum_{i=o}^{t_o-1} \alpha^i \varepsilon_{t-i}$$

In equation (1) we can permit current-dated exogenous variables and random disturbances to affect X_t without this requiring any significant alteration in the solution method. Replace (1) by (1').

$$(1') \quad \begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_{t+1}$$ [1/]

The reduced form solution is, using the notation of Blanchard and Kahn,

$$P_t = - C_{22}^{-1} C_{21} X_t - C_{22}^{-1} \sum_{i=o}^{\infty} J_2^{-i-1} [C_{21} \gamma_1 + C_{22} \gamma_2] \; {}_tZ_{t+i+1}$$

$X_t = X_o$ for $t = o$

$$X_t = B_{11} J_1 B_{11}^{-1} X_{t-1} + \gamma_1 Z_t$$
$$- [B_{11} J_1 C_{12} + B_{12} J_2 C_{22}] C_{22}^{-1} \sum_{i=o}^{\infty} J_2^{-i-1} [C_{21} \gamma_1 + C_{22} \gamma_2] \; {}_{t-1}Z_{t+i}$$

for $t > o$

Where

$A = C^{-1} JC$

$$J = \begin{bmatrix} J_1 & O \\ O & J_2 \end{bmatrix} ; \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} ; \quad C^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$ [2/]

[1/] For this to make economic sense, the components of γ_2 corresponding to value of the forcing variables in period $t + 1$ should be zero: P_t cannot be a function of actual future realizations of Z, only of anticipated future realizations.

[2/] J_1 is the nxn diagonal matrix whose diagonal elements are the n stable characteristic roots of A; J_2 is the mxm diagonal matrix whose diagonal elements are the unstable characteristic roots of A. C is the matrix whose columns are the right-characteristic vectors of A. A is assumed to be diagonalizable.

In a recent paper Chow [1981] proposed the following less restrictive definition of a predetermined variable: X_t is predetermined i.f.f. $X_t \equiv E\left(X_t|\Omega_t\right)$. This definition too leads to a characterization that may contradict economic sense and may not fit into the solution method proposed by Blanchard and Kahn. Consider the following model where ε_t again represents white noise.

(5) $$y_t = \beta E\left(y_{t+1}|\Omega_t\right) + \delta Z_t + \varepsilon_t$$

Assume that Ω_t includes the true structure of the model and complete contemporaneous information on the endogenous (y) and exogenous (Z) variables. This implies that $y_t = E\left(y_t|\Omega_t\right)$, yet the natural economic interpretation of y is that of a forward-looking, non-predetermined variable. Assuming that $\lim_{n\to\infty} \beta^n E\left(y_{t+n}|\Omega_t\right) = 0$ and that the infinite sum on the right-hand side of (6) exists, the forward-looking solution for (5) is:

(6) $$y_t = \delta Z_t + \varepsilon_t + \delta \sum_{i=1}^{\infty} \beta^i E\left(Z_{t+i}|\Omega_t\right)$$

y_t is a non-predetermined variable because it is a function of expectations formed at time t : it can "jump" in response to "news".

Apart from classifying certain non-predetermined variables as predetermined, Chow's proposed definition also classifies certain predetermined variables as non-predetermined. Consider again equation (3). Whether or not $E\left(y_{t+1}|\Omega_{t+1}\right) = y_{t+1}$ depends on the information set. If agents don't observe y_{t+1} exactly until period t+2 and observe in period t+1 some noisy function of y_{t+1}, say $\zeta_{t+1} = \mu y_{t+1} + w_{t+1}$, they are faced with a signal extraction problem. Let $E(w_t) = E(\varepsilon_t) = 0$, $E(w_t^2) \equiv \sigma_w^2$, $E(\varepsilon_t^2) \equiv \sigma_\varepsilon^2$, $E(\varepsilon_t w_t) = \sigma_{\varepsilon w}$ for all t and $E(\varepsilon_t w_s) = 0$ for all t and s, $s \neq t$.

(7) $E\left(y_t|\Omega_t\right) = E\left(y_t|y_{t-1}, \zeta_t\right) = \alpha y_{t-1} + \frac{\mu\sigma_\varepsilon^2+\sigma_{\varepsilon w}}{\mu^2\sigma_\varepsilon^2+\sigma_w^2+2\mu\sigma_{\varepsilon w}}\left[\mu\varepsilon_t+w_t\right]$

$\neq y_t = \alpha y_{t-1} + \varepsilon_t$

Yet y is still a predetermined variable: y_t is not a function of expectations, formed at t, of future endogenous and/or exogenous variables.

It will be clear from this discussion what the proposed alternative definition of a predetermined and non-predetermined variable is.

Definition

X_t is predetermined i.f.f. X_t is not a function of expectations, formed at t, of future endogenous and/or exogenous variables.

P_t is non-predetermined i.f.f. P_t is a function of expectations, formed at t, of future endogenous and/or exogenous variables.

Note that this fits the general solution method proposed by Blanchard and Kahn for their model. Their final form solution for X_t depends on expectations of future exogenous variables formed in period t-1 and before. The final form solution for P_t depends on expectations of future exogenous variables formed in period t and before.

Consider the following three examples:

(8a) $y_t = \alpha y_{t-1} + \beta E\left(y_t|\Omega_{t-1}\right) + \gamma Z_t + \varepsilon_t$

(8b) $y_t = \alpha y_{t-1} + \beta E\left(y_{t+1}|\Omega_t\right) + \gamma Z_t + \varepsilon_t$

(8c) $y_t = \alpha y_{t-1} + \beta E\left(y_{t+1}|\Omega_{t-1}\right) + \gamma Z_t + \varepsilon_t$

Assume Ω_t includes the correct structure of the model and all current and past values of the exogenous and endogenous variables. According to our definition, y in (8a) is a predetermined variable. While it cannot be put in Blanchard and Kahn's "first order" form (1) its reduced form solution is easily found to be

$$\text{(9a)}\quad y_t = \alpha(1-\beta)^{-1} y_{t-1} + \gamma Z_t + \beta(1-\beta)^{-1} \gamma E\left(Z_t | \Omega_{t-1}\right) + \varepsilon_t$$

In the structural form (8a) y_t is function only of past expectations of the current endogenous variable. In the reduced form (9a) y_t is a function only of past expectations of the current exogenous variables. y_t is predetermined.

In equation (8b) y is a non-predetermined variable. It can be put in Blanchard and Kahn's "first order" form:

$$\begin{bmatrix} y_t \\ E\left(y_{t+1} | \Omega_t\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta^{-1}\alpha & \beta^{-1} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\beta^{-1}\gamma & -\beta^{-1} \end{bmatrix} \begin{bmatrix} Z_t \\ \varepsilon_t \end{bmatrix}$$

The forward-looking reduced form solution is (assuming stability i.e. one stable and an unstable root in $\lambda^2 + \beta^{-1}\alpha\lambda - \beta^{-1} = 0$)

$$\text{(9b)}\quad y_t = \pi_1 y_{t-1} + \pi_2 Z_t + \pi_3 \varepsilon_t + \sum_{i=1}^{\infty} \eta_i E\left(Z_{t+i} | \Omega_t\right)$$

where

$$\pi_1 = \alpha\left(1-\beta\pi_1\right)^{-1} \;;\quad \pi_2 = \left(1-\beta\pi_1\right)^{-1}\gamma \;;\quad \pi_3 = \left(1-\beta\pi_1\right)^{-1}$$

$$\eta_1 = \left(1-\beta\pi_1\right)^{-2} \beta\gamma \;;\quad \eta_i = \left(1-\beta\pi_1\right)^{-1} \beta\eta_{i-1} \;,\; i \geq 2 \;.$$

In the structural form (8b), y_t is a function of current expectations of a future endogenous variable. In the reduced form (9b) y_t is a function of current expectations of all future values of the forcing variables. y_t is non-predetermined. It can make discrete jumps in response to "news" at time t.

In equation (8c) y is a predetermined variable, in spite of the presence of forward-looking expectations. It can be put in Blanchard and Kahn's "first order" form in spite of the presence of lagged expectations of a future variable.

$$\begin{bmatrix} y_{t+1} \\ E\left(y_{t+2}|\Omega_t\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta^{-1}\alpha & \beta^{-1} \end{bmatrix} \begin{bmatrix} y_t \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\beta^{-1}\gamma & -\beta^{-1} \end{bmatrix} \begin{bmatrix} z_{t+1} \\ \varepsilon_{t+1} \end{bmatrix}$$

Assuming stability, the forward-looking reduced form solution is

$$\text{(9c)} \quad y_t = \pi_1 y_{t-1} + \pi_2 z_t + \pi_3 \varepsilon_t + \sum_{i=0}^{\infty} \eta_i E\left(z_{t+i}|\Omega_{t-1}\right)$$

where

$$\pi_1 = \alpha\left[1-\pi_1\beta\right]^{-1} \; ; \quad \pi_2 = \gamma \; ; \quad \pi_3 = 1 \; ; \quad \eta_0 = \pi_1 \beta\left(1-\pi_1\beta\right)^{-1} \gamma \; ;$$

$$\eta_1 = \left(1-\pi_1\beta\right)^{-2} \beta\gamma \; ; \quad \eta_i = \left(1-\beta\pi_1\right)^{-1}\beta\eta_{i-1} \qquad i \geq 2$$

In the structural form (8c) y_t is a function of past expectations of a future endogenous variable. In the reduced form (9c) y_t is a function of past expectations of all future values of the forcing variables. While y_t is a function of anticipated future events, it is not a function of current anticipations of future events. y_t does not respond to current "news". It is non-predetermined.

Conclusion

The current values of non-predetermined variables, unlike predetermined variables, are functions of current anticipations of future values of endogenous and/or exogenous variables. This property is crucial both for the mathematical derivation of solution algorithms for rational expectations models and for their economic interpretation.

References

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