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A NOTE ON THE SOLUTION OF A TWO-POINT BOUNDARY VALUE PROBLEM FREQUENTLY ENCOUNTERED IN RATIONAL EXPECTATIONS MODELS

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ABSTRACT

The paper analyses a class of two-point boundary value problem for systems of linear differential equations with constant coefficients. The boundary conditions are expressed as linear restrictions on the state vector at an initial time and at a finite terminal time. This is applicable even if the terminal conditions involve the asymptotic convergence of the system to steady-state equilibrium, as is frequently the case in economic applications. It is also a suitable format for numerical applications using existing computer routines. The case in which there are more stable eigenvalues than predetermined state variables is also considered. An example involving a small open economy macroeconomic model is used to illustrate the analysis.

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I Introduction

This note analyses the solution of systems of linear differential equations with constant coefficients for which boundary conditions are given at two points in time. Two-point boundary value problems were originally developed for engineering and natural science applications. They first became familiar to economists studying continuous time dynamic optimization problems using Hamiltonian methods. To obtain the control variables as functions of time one has to solve for the time paths of the co-state variables. This means solving a two-point boundary value problem with initial boundary conditions for the state variables and terminal boundary conditions for the co-state variables.

In recent years the scale of application has been extended greatly through the widespread use of the rational expectations assumption and the efficient markets hypothesis in macroeconomics. Analytical treatment has generally been limited to systems of two dynamic equations, which permit heuristic graphical analysis, although a few studies of systems with three or more dynamic equations exist (Calvo [1979], Obstfeld [1980], Dixit [1980]). Also recently, Lipton, Poterba, Sachs and Summers [1980], have presented algorithms for numerically calculating the saddlepoint paths that represent the solution of many rational expectations models. As their methods apply to non-linear as well as linear models, they have the advantage of a considerable degree of generality. This note is much less ambitious in that it considers only systems of linear differential equations (see Blanchard and Kahn [1980] for a treatment of linear difference equations systems). This means that, provided the state transition matrix is of full rank and provided n linearly independent boundary conditions are given, there are no problems of non-uniqueness. [1/] Non-uniqueness problems are especially baffling in rational expectations models with "forward-looking" state variables such as asset prices determined in efficient markets, which incorporate and reflect information about the current and anticipated future behaviour of the exogenous variables. It isn't even clear conceptually how one would approach rational expectations models whose solutions are characterized by sequences of multiple temporary or momentary equilibria.

The main contribution of this note is in the treatment of the boundary conditions for the "forward-looking" state variables i.e. those that cannot be treated as predetermined. Typically, in economic applications, the terminal conditions that complement the initial conditions for the predetermined or "backward-looking" variables involve an infinite time horizon. The most common assumption, considered in this paper, is that the system converges (asymptotically) to a steady state equilibrium. Numerical algorithms aren't very sympathetic to terminal conditions at $t = +\infty$. The usual response of practitioners is to hope that a large but finite time horizon will approximate adequately the infinite horizon. In Section IV of this note we consider the case where the number of predetermined variables equals the number of stable eigenvalues (those with negative real parts) and the number of forward-looking or jump variables equals the number of unstable eigenvalues (those with positive real parts). Conditions are given under which a set of linear restrictions on the state vector at some finite future time, $t = t_f$, is exactly equivalent to the condition of asymptotic convergence to the steady-state equilibrium. The initial conditions, at $t = t_o \leq t_f$, on the predetermined state variables can of course also be expressed as a set of linear restrictions on the state vector, at $t = t_o$. In Section V I show how the boundary conditions for a third kind of state variable, which combines aspects of the pure backward-looking and forward-looking state variables, can be expressed as the sum of a set of initial conditions at $t = t_o$ and a linear function of the values of the pure jump variables at $t = t_o$. Other ways of obtaining a unique solution when there are "too many" stable eigenvalues are also considered.

This specification of the boundary conditons has two advantages. First, solutions for linear two-point boundary value problems exist when the boundary conditions are expressed as sets of linear restrictions on the state vector at an initial date and at a finite terminal date. Two such methods, the "method of adjoints" and the "forward sweep" solution are sketched briefly in Section III. Second, numerical algorithms exist for two-point boundary value problems that require the boundary conditions to be entered this way. An example is the NAG Routine DO2AFF (NAG [1977]).

II. The state-space representation of differential equation systems

The general structural form of the differential equation system we shall analyse is given in equations (1) and (2). D if the differential operator, i.e. $D\,x(t) \equiv \frac{d}{dt} x(t)$. All vectors are column vectors. a^T denotes the transpose of a.

(1) $\Gamma_1\, x(t) + \Gamma_2\, D\, x(t) + \Gamma_3\, y(t) + \Gamma_4\, z(t) = 0$

(2) $\Gamma_5\, x(t) + \Gamma_6\, y(t) + \Gamma_7\, z(t) = 0$.

x is an n vector of state variables. y(t) is a q vector of output variables. z(t) is an m vector of exogenous or forcing variables. Γ_i , i = 1 , ... 7 are matrices with constant coefficients. If Γ_2 and Γ_6 are of full rank, (1) and (2) have the state-space or dynamic reduce form representation of (1') and (2').

(1') $D\,x(t) = A\,x(t) + B\,z(t)$

(2') $y(t) = C\,x(t) + D\,z(t)$

where

(3a) $A = -\Gamma_2^{-1}\left[\Gamma_1 - \Gamma_3\,\Gamma_6^{-1}\,\Gamma_5\right]$

(3b) $B = \Gamma_2^{-1}\left[\Gamma_3\,\Gamma_6^{-1}\,\Gamma_7 - \Gamma_4\right]$

(3c) $C = -\Gamma_6^{-1}\,\Gamma_5$

(3d) $D = -\Gamma_6^{-1}\,\Gamma_7$

We shall henceforth work with the state-space representation of (1') and (2'). Given a solution for x(t) using (1') and a set of boundary conditions for x , y(t) can be obtained simply by repeatedly solving a system of linear equations. We therefore focus on (1').

III Solutions of the two-point boundary value problem for linear differential equations with linear boundary conditions.

a) The method of adjoints

Consider the linear differential equation system (1'), reproduced again for ease of reference, with n linear boundary conditions as given in (4).

(1') $D\,x(t) = A\,x(t) + B\,z(t)$

(4) $\sum_{i=1}^{n} \mu_{ji}\, x_i(t_o) + \sum_{i=1}^{n} \nu_{ji}\, x_i(t_f) = \rho_j \qquad j = 1, 2, \ldots, n\ ; \quad t_f \geq t_o$

x_i , i = 1, 2, ..., n is the i^{th} component of the state vector x. The μ_{ji} , ν_{ji} and ρ_j are known constants. Equation (12) can be rewritten as:

(4') $M\,x(t_o) + N\,x(t_f) = R$

M is an nxn matrix, $M \equiv \{\mu_{ji}\}$; N is an nxn matrix, $N \equiv \{\nu_{ji}\}$ and R is an n-vector, $R \equiv (\rho_1, \ldots, \rho_j, \ldots, \rho_n)^T$.

Consider the adjoint system to (1').

(5) $D\,\psi(t) = -\,A^T\,\psi(t)$

Integrate the adjoint equations backward from $t = t_f$, once for each $x_i(t_f)$ appearing in (4), using as the terminal boundary conditions:

(6) $\psi_i^{(j)}(t_f) = \nu_{ji} \qquad i, j = 1, 2, \ldots, n$.

$\psi_i^{(j)}(t_f)$ is the i^{th} component at $t = t_f$ for the j^{th} backward integration of the adjoint equation. Thus, if ν_j^T denotes the transpose of the j^{th} row of N in equation (4'), we solve

(7) $$\psi^{(j)}(t) = e^{-(t-t_f)A^T} \nu_j^T \qquad j = 1, 2, \ldots, n \; .$$

Note that for any matrix W

(8) $$e^W \equiv I + W + \frac{1}{2!} W^2 + \frac{1}{3!} W^3 + \ldots .$$

Setting $t = t_o$ in (7) we obtain $\psi^j(t_o)$.

The fundamental identity for the method of adjoints is:

(Roberts and Shipman [1972, pp 17-22 and pp 39-40])

(9) $$\sum_{i=1}^{n} \psi_i^{(j)}(t_f)\; x_i(t_f) - \sum_{i=1}^{n} \psi_i^{(j)}(t_o)\; x_i(t_o) = \int_{t_o}^{t_f} \sum_{i=1}^{n} \psi_i^{(j)}(t)\; b_i z(t) dt$$

$$j = 1, 2, \ldots, n.$$

b_i is the i^{th} row of the matrix B, i.e.

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

Substituting for $\psi_i^j(t_f)$ from (6) into (9) and using (4) yields:

$$\rho_j - \sum_{i=1}^{n} \mu_{ji}\, x_i(t_o) - \sum_{i=1}^{n} \psi_i^{(j)}(t_o)\, x_i(t_o) = \int_{t_o}^{t_f} \sum_{i=1}^{n} \psi_i^{(j)}(t)\, b_i z(t) dt$$

$$j = 1, 2, \ldots, n .$$

or

$$(10) \quad \sum_{i=1}^{n} \left(\mu_{ji} + \psi_i^{(j)}(t_o)\right) x_i(t_o) = \rho_j - \int_{t_o}^{t_f} \sum_{i=1}^{n} \psi_i^{(j)}(t)\, b_i z(t) dt$$

$$j = 1, 2, \ldots, n .$$

Equation (10) constitutes a set of n equations in the n unknowns $x_i(t_o)$, $i = 1, 2, \ldots, n$. If they are linearly independent they will yield a unique solution for $x(t_o)$. Given $x(t_o)$ equation (1') can be solved as a standard initial value problem. The solution is:[2/]

$$(11) \quad x(t) = e^{(t-t_o)A} x(t_o) + \int_{t_o}^{t} e^{(t-T)A} Bz(T) dT$$

Note that this problem can be solved in one "pass", i.e. without iterations.

In all macroeconomic applications that I am aware of, the boundary conditions (4) and (4') specialize to the separable case of linear boundary conditions at each boundary:

$$(4a) \quad \sum_{i=1}^{n} \mu_{ji}\, x_i(t_o) = \delta_j \qquad j = 1, 2, \ldots, n_1 \qquad n_1 \leqslant n$$

and

$$(4a') \quad \sum_{i=1}^{n} \nu_{ji}\, x_i(t_f) = \varepsilon_j \qquad j = 1, 2, \ldots, n - n_1$$

We now integrate the adjoint equations (5) backwards $n - n_1$ times (one for each specified terminal boundary condition in (4a')), using as terminal conditions:

(6a) $\psi_i^{(j)}(t_f) = \nu_{ji}$ $\quad j = 1, 2, \ldots, n - n_1 ; \quad i = 1, 2, \ldots, n .$

Consider the first summation in (9). Using (4a') and (6a) we obtain:

$$\sum_{i=1}^{n} \psi_j^{(j)}(t_f)\, x_i(t_f) = \sum_{i=1}^{n} \nu_{ji}\, x_i(t_f) = \varepsilon_j \qquad j = 1, 2, \ldots, n - n_1 .$$

Equation (9) can now be rewritten as

(10a) $$\sum_{i=1}^{n} \psi_i^{(j)}(t_o)\, x_i(t_o) = \varepsilon_j - \int_{t_o}^{t_f} \sum_{i=1}^{n} \psi_i^{(j)}(t)\, b_i z(t)dt \qquad j = 1, 2, \ldots, n - n_1 .$$

The n_1 initial linear boundary conditions in (4a) and the $n - n_1$ equations (10a) provide n equations in the n unknowns $x_i(t_o)$ $\quad i = 1, 2, \ldots, n$. If they are linearly independent they will determine a unique initial value for $x(t)$ at $t = t_o$.

b) The "forward-sweep" method

If the boundary conditions are separable into a set of linear restrictions on $x(t_o)$ and a set of linear restrictions on $x(t_f)$, as in (4a, a'), the following solution method can be applied (Bryson and Ho [1975, p. 176]).

(1') $D\, x(t) = A\, x(t) + B\, z(t)$

(12a) $K_1\, x(t_o) = a$

(12b) $K_2\, x(t_f) = b$.

K_1 is an $n_1 x n$ matrix, K_2 an $(n-n_1) x n$ matrix, a an n_1 vector and b an $n-n_1$ vector. $n_1 \leq n$. It is assumed that K_1 and K_2 contain n linearly independent restrictions, i.e. that the nxn matrix $\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ is invertible.

A "forward-sweep" solution to (1'), (12a) and (12b) is obtained by considering the system of equation (13).

$$(13) \quad K_1 \, x(t) = S(t) \, K_2 \, x(t) + m(t) \ .$$

S(t) is an $n_1 x(n-n_1)$ matrix and m(t) an n vector. They are obtained by solving the initial value problem of equations (14a,b) and (15a,b).

$$(14a) \quad D \, S(t) = F_1 \, S(t) - S(t) \, F_4 - S(t) \, F_3 \, S(t) + F_2$$

$$(14b) \quad S(t_o) = 0$$

$$(15a) \quad D \, m(t) = \left(F_1 - S(t) \, F_3\right) m(t) + \left(K_1 - S(t) \, K_2\right) B \, z(t)$$

$$(15b) \quad m(t_o) = a$$

where

$$(16) \quad \left[\begin{array}{c|c} F_1 & F_2 \\ \hline F_3 & F_4 \end{array}\right] = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} A \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}^{-1}$$

Equations (14a,b) and (15a,b) represent a non-linear single point boundary value problem or initial value problem. Using standard initial value problem algorithms, they can be integrated forward in time from the initial conditions for S and m at $t = t_o$ given in (14b) and (15b).[3/] At $t = t_f$ this yields (using (13) and 12b))

$$(17) \quad K_1\, x(t_f) = S(t_f)b + m(t_f)$$

$$(12b) \quad K_2\, x(t_f) = b$$

(17) and (12b) represent n linearly independent equations that uniquely determine $x(t_f)$. Denote this solution for $x(t_f)$ by $\bar{x}(t_f)$. The two-point boundary value problem has again been transformed into a single-point boundary value problem. We can now solve for $x(t)$, $t_o \leq t \leq t_f$ by integrating (1') backwards from $\bar{x}(t_f)$. The solution is given by:

$$(11') \quad x(t) = e^{(t-t_f)A}\,\bar{x}(t_f) - \int_t^{t_f} e^{(t-T)A}\, BZ(T)\,dT \qquad t_f \geq t \geq t_o .$$

Note that the boundary conditions (12a) and (12b) are a special case of the boundary conditions in (4) and (4'),as (12a, b) can be rewritten as

$$\begin{bmatrix} K_1 \\ 0_{(n-n_1)xn} \end{bmatrix} x(t_o) + \begin{bmatrix} 0_{n_1xn} \\ K_2 \end{bmatrix} x(t_f) = \begin{bmatrix} a \\ b \end{bmatrix}$$

0_{ixj} denotes the ixj zero matrix.

In Section IV we show how, in models whose state variables are either pure backward-looking (or predetermined) or pure forward-looking (or jump) variables, the boundary conditions can be expressed as in (12a) and (12b) and therefore also as in (4) or (4'). In Section V it is shown how in models containing one or more "mixed" backward-looking and forward-looking state variables, the boundary conditions can again be expressed either in terms of (12a, b) or in terms of (4) or 4').

IV The representation of the boundary conditions in rational expectations models as linear restrictions on the state vector: the number of predetermined variables equals the number of stable eigenvalues.

In many rational expectations models the vector of state variables, x , can be partitioned into a set of n_1 backward-looking or predetermined variables $_1x$ and a set of $n - n_1$ forward-looking or jump variables, $_2x$, i.e.

$$(18)\quad x = \left({}_1x^T \;\; {}_2x^T\right)^T$$

$_1x$ is an n_1 vector, $_2x$ an $(n - n_1)$ vector. Predetermined variables are differentiable functions of time; their values are given at a point in time by past history. The stock of physical capital, the stock of net claims on the rest of the world in an open economy and, in some Keynesian models, the money wage and the price level are examples.
The boundary conditions for these n_1 predetermined variables take the form of the assignment of initial values at $t = t_o$.

(19) ${}_1x\ (t_o) = {}_1\bar{x}\ (t_o)$

Jump variables are variables whose values are not given at a point in time by past history. In rational expectations models prices determined in efficient markets such as financial asset prices often fall into this category. If the price of an asset depends on its expected rate of change, its current value can be obtained by solving for the entire expected future path of the price. The current price can therefore make discrete jumps at a point in time in response to "news", that is in response to new information about the current or future behaviour of the forcing variables. This means that the left-hand side and right-hand side time derivatives of ${}_2x(t)$ need not coincide and that the left-hand side time derivative need not be bounded. The terminal boundary conditions for these "jump" variables generally take the form of a transversality condition: the system is required to converge asymptotically to its steady state equilibrium.[4/] We shall make these intuitive notions more precise as follows.

<u>Assumption 1</u>. After some point in time $t_f \geq t_o$, the forcing variables are constant. If this weren't so, no stationary equilibrium would exist for x . Thus, there exists a $t_f \geq t_o$ such that

(20) $z(t) = \bar{\bar{z}}$ for $t \geq t_f$.

Given (20) there exists a stationary equilibrium value of x given by

$$0 = A\ x + B\ \bar{\bar{z}} \quad .$$

Assumption 2. A is invertible. There is no significant loss of generality here. Given Assumption 2, the steady-state value of x is given by

$$(21)\quad \bar{x} = - A^{-1} B \bar{\bar{z}} .$$

The second boundary condition, complementing (19) that we impose is

$$(22)\quad \lim_{t\to\infty} x(t) = - A^{-1} B \bar{\bar{z}} .$$

Note that this does not rule out convergence to the steady state in finite time.

We will now derive conditions under which (19) and (22) provide n linearly independent boundary conditions on x, thus guaranteeing a unique solution. At the same time we show how the boundary (22) condition at $t = \infty$ can be transformed into a boundary condition at $t_f < \infty$, which is of the general form given in (12a, b).

Assumption 3. A has n_1 eigenvalues with negative real parts and $n - n_1$ eigenvalues with positive real parts.

This condition that the number of stable eigenvalues equal the number of predetermined variables and the number of unstable eigenvalues equal the number of jump variables will turn out to be useful in guaranteeing a unique solution to (1'), (19) and (22).

Assumption 4. A has n distinct eigenvalues. This assumption greatly simplifies the analysis.[5/]

If the nxn matrix A has n distinct eigenvalues it can be reduced to diagonal form by means of a similarity transformation. Let Λ be the diagonal matrix whose diagonal elements are the eigenvalues of A , i.e.

(23)
$$\Lambda = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_i & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix}$$

The λ_i , i = 1 , ..., n are the solutions to the characteristic equation

$$|A - \lambda I| = 0$$

Let V be the nxn matrix whose columns are the right eigenvectors of A , i.e.

(24) $V = [v_1 \ldots v_i \ldots v_n]$

v_i is the right eigenvector corresponding to λ_i , i.e. it is obtained by solving.

(25) $Av_i = \lambda_i v_i$

It then follows that

(26a) $A = V \Lambda V^{-1}$

or

(26b) $\Lambda = V^{-1} A V$.

Let

(27a) $p = V^{-1} x$

or

(27b) $x = V p$

Then (1') can be transformed into

(28) $D p(t) = \Lambda p(t) + V^{-1} B z(t)$.

Without loss of generality Λ and p can be rearranged and conformably partitioned as follows:

(29a)
$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

(29b) $p = \begin{bmatrix} p_1^T & p_2^T \end{bmatrix}^T$

Λ_1 is the $n_1 \times n_1$ diagonal matrix whose diagonal elements are the eigenvalues of A with negative real parts. Λ_2 is the $(n - n_1)x(n-n_1)$ diagonal matrix whose diagonal elements are the eigenvalues of A that have positive real parts. p_1 is an n_1 vector and p_2 an $(n - n_1)$ vector .

Let

(30) $W(t) \equiv V^{-1} B z(t)$

We partition $W(t)$ into its first n_1 elements and its last $n - n_1$ elements

(31a) $$W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$$

Thus W_1 and W_2 are n_1, respectively $n - n_1$ vectors defined by:

(31b) $W_1(t) = \begin{bmatrix} I_{n_1} & O_{n_1 x(n-n_1)} \end{bmatrix} V^{-1} B z(t)$

and

(31c) $W_2(t) = \begin{bmatrix} O_{(n-n_1)xn_1} & I_{n-n_1} \end{bmatrix} V^{-1} B z(t)$.

I_r is the identity matrix of order r .

Using (29a, b) and (31a), (28) can be rewritten as

(32a) $D p_1(t) = \Lambda_1 p_1(t) + W_1(t)$

(32b) $D p_2(t) = \Lambda_2 p_2(t) + W_2(t)$

For $t \geq t_f$ (32b) becomes, using (31c) and (20)

(33) $$D\, p_2(t) = \Lambda_2\, p_2(t) + \left[O_{(n-n_1)xn_1} \quad I_{n-n_1} \right] V^{-1} B\, \bar{\bar{z}}$$

All the diagonal elements of the diagonal matrix Λ_2 have positive real parts. Therefore, for the system to converge to its steady state equilibrium it is necessary that $p_2(t_f)$ assumes the value required to ensure that $D\, p_2(t_f) = 0$. At $t = t_f$ the system must be on the n_1-dimensional stable manifold, the subspace spanned by the eigen vectors associated with the eigen values that have negative real parts, where it will stay for all $t \geq t_f$, if the system is to converge to the steady-state equilibrium. The asymptotic convergence condition in (22) can before be replaced by a set of linear restrictions on $p_2(t)$ at $t = t_f$.

(34) $$p_2(t_f) = - \Lambda_2^{-1} \left[O_{(n-n_1)xn_1} \quad I_{n-n_1} \right] V^{-1} B\, \bar{\bar{z}}$$

We can transform (34) into an equivalent set of linear restrictions on $x(t)$ at $t = t_f$. From (27a) and (29b) we see that

(35) $$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = V^{-1} x$$

Partition V^{-1} into its first n_1 and its last $n - n_1$ rows, as follows

(36) $$V^{-1} = \begin{bmatrix} \left(V^{-1}\right)_1 \\ \left(V^{-1}\right)_2 \end{bmatrix}$$

$\left(V^{-1}\right)_1$ is an n_1xn matrix and $\left(V^{-1}\right)_2$ an $(n-n_1)xn$ matrix.

From (35) and (36) we get

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \left(V^{-1}\right)_1 \\ \left(V^{-1}\right)_2 \end{bmatrix} x$$

or

$$(37) \quad p_2(t) = \left(V^{-1}\right)_2 x(t)$$

At $t = t_f$, using (34) and (37) we then have:

$$(38) \left(V^{-1}\right)_2 x(t_f) = -\Lambda_2^{-1} \begin{bmatrix} O_{(n-n_1)xn_1} & I_{n-n_1} \end{bmatrix} V^{-1} B \bar{\bar{z}}$$

This is a set of linear restrictions on $x(t_f)$ of the type considered in equation (12b). It is of course trivial to express the initial conditions on the predetermined variables ${}_1x$, given in (19) as a set of linear restrictions on x at $t=t_o$. (19) can be expressed in the format of equation (4) as follows:

$$(39) \begin{bmatrix} I_{n_1} & O_{n_1 x(n-n_1)} \end{bmatrix} x(t_o) = {}_1\bar{x}\,(t_o)$$

For any path of $z(t)$, $t_o \leq t < t_f$ (with z an integrable function of time) the solution of the two-point boundary value problem $D\,x(t) = A\,x(t) + B\,z(t)$ with boundary conditions (38) and (39) can now be found for the interval $t_o \leq t \leq t_f$ using the method of adjoints or the forward sweep method of Section III.

Note that if A can be diagonalized, as we assume, then

(40a) $e^{tA} = V\, e^{t\Lambda}\, V^{-1}$

where

(40b)
$$e^{t\Lambda} = \begin{bmatrix} e^{\lambda_1 t} & & & & 0 \\ & \ddots & & & \\ & & e^{\lambda_i t} & & \\ & & & \ddots & \\ 0 & & & & e^{\lambda_n t} \end{bmatrix}$$

For a unique solution to exist, the nxn matrix Ω defined by

(41)
$$\Omega = \begin{bmatrix} I_{n_1} & O_{n_1 x(n-n_1)} \\ \left(V^{-1}\right)_2 & \end{bmatrix}$$

should be of full rank. In the forward sweep solution method of Section IIIb, this shows up as the requirement that the matrix $\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$, where K_1 and K_2 are defined in equations (12a,b), be invertible. (Partition $\left(V^{-1}\right)_2$) into its first n_1 columns and its last $n - n_1$ columns:

(42) $\left(V^{-1}\right)_2 = \left[\left(V^{-1}\right)_{21}\ \left(V^{-1}\right)_{22}\right]$

$\left(V^{-1}\right)_{21}$ is an $(n-n_1)xn_1$ matrix and $\left(V^{-1}\right)_{22}$ an $(n-n_1)x(n-n_1)$ matrix. Substituting (42) into (41) we get

(43)
$$\Omega = \begin{bmatrix} I_{n_1} & O_{n_1 x(n-n_1)} \\ \left(V^{-1}\right)_{21} & \left(V^{-1}\right)_{22} \end{bmatrix}$$

Uniqueness therefore requires that $\left(V^{-1}\right)_{22}$ be of full rank, $n - n_1$.

To solve for $x(t)$ for $t > t_f$ we use the condition that for $t \geq t_f$ $x(t)$ lies on the stable manifold. Thus

$$(44)\quad x(t) = e^{(t-t_f)A}\left(x(t_f) + A^{-1} B \bar{\bar{z}}\right) - A^{-1} B \bar{\bar{z}} \ , \quad t \geq t_f$$

Given $x(t_f)$, this is a standard initial value problem.

Nothing can be said about the behaviour of $x(t)$ for $t < t_o$, without further information. Presumably the reason for choosing t_o as the initial date, is that at that moment new information became available that altered expectations concerning the current and future behaviour of the forcing variables. To determine $x(t)$ for $t < t_o$, we need another initial condition for ${}_1x(t)$ at, say, $t_o' < t_o$. Given the entire anticipated (as of $t = t_o'$) path of the forcing variables, we can use the two point boundary value method outlined in Sections III and IV to solve for $x(t)$, $t_o' \leq t < t_o$. At t_o , when the "news" arrives, the rest of this solution becomes irrelevant, and a new two-joint boundary value is solved, corresponding to the new perceptions of $z(t)$, $t \geq t_o$. This procedure is repeated every time new information leads to revisions in the anticipated future trajectory of $z(t)$. The initial values of the predetermined variables ${}_1x$ at t_o are of course inherited from the past; ${}_1x$ does not make discrete jumps in response to new information: $\lim_{\substack{t \to t_o \\ t < t_o}} {}_1x(t) = {}_1x(t_o)$. The jump variables ${}_2x(t)$, while right-continuous everywhere, can be discontinuous at points such as t_o when new information becomes available. We do not require $\lim_{\substack{t \to t_o \\ t < t_o}} {}_2x(t) = {}_2x(t_o)$; this is of course how the two-point boundary value problem arises in the first place. It is often convenient to assume that before $t = t_o$, the system was in a steady state equilibrium, corresponding to a constant path $\bar{z}$ of the forcing variables, i.e.

$$(45) \quad x(t) = \begin{bmatrix} {}_1x(t) \\ {}_2x(t) \end{bmatrix} = A^{-1} B \bar{z} \qquad t < t_o$$

Consistency then of course requires that ${}_1x(t_o)$ be equal to ${}_1x(t)$, $t < t_o$.

In a number of economic applications, $t_o = t_f$. This represents an unanticipated and immediate once and for all change in the values of exogenous variables: the change in the future path of z occurs at the instant that it is first anticipated. Since z immediately assumes its new steady state value, $\bar{\bar{z}}$, at $t = t_o$, the calculation of $x(t_f) = x(t_o)$ is particularly simple (see Dixit [1980] and Buiter and Miller [1980]). Using (38) and (42) we obtain

$$\left(V^{-1}\right)_{21} {}_1\bar{x}(t_o) + \left(V^{-1}\right)_{22} {}_2x(t_o) = - \Lambda_2^{-1} \left[0_{(n-n_1)xn_1} \quad I_{n-n_1} \right] V^{-1} B \bar{\bar{z}}$$

or

$$(46) \quad {}_2x(t_o) = - \left[\left(V^{-1}\right)_{22} \right]^{-1} \left\{ \left(V^{-1}\right)_{21} {}_1\bar{x}(t_o) + \Lambda_2^{-1} \left[0_{(n-n_1)xn_1} \quad I_{n-n_1} \right] V^{-1} B \bar{\bar{z}} \right\}$$

$x(t_f) = x(t_o)$ is obtained from (46), given the initial value ${}_1\bar{x}(t_o)$. The rest of the solution is given in (44).

Note that we can write Dx = Ax + Bz as

$$Dx = V \Lambda V^{-1} x + Bz$$

Partitioning V, Λ and V^{-1} conformably this can in turn be written as

$$Dx = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{21} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} (V^{-1})_{11} & (V^{-1})_{12} \\ (V^{-1})_{21} & (V)^{-1}_{22} \end{bmatrix} x + Bz$$

or

$$Dx(t) = \begin{bmatrix} V_{11}\Lambda_1 (V^{-1})_{11} & V_{11}\Lambda_1 (V^{-1})_{12} \\ V_{21}\Lambda_1 (V^{-1})_{11} & V_{21}\Lambda_1 (V^{-1})_{12} \end{bmatrix} x(t) + \begin{bmatrix} V_{12}\Lambda_2 (V^{-1})_{21} & V_{12}\Lambda_2 (V^{-1})_{22} \\ V_{22}\Lambda_2 (V^{-1})_{21} & V_{22}\Lambda_2 (V^{-1})_{22} \end{bmatrix} x(t) + Bz(t)$$

Using (46) this simplifies for $t \geq t_f$, when the system is on the stable manifold, to:

$$(47) \quad Dx(t) = \begin{bmatrix} V_{11}\Lambda_1 (V^{-1})_{11} & V_{11}\Lambda_1 (V^{-1})_{12} \\ V_{21}\Lambda_1 (V^{-1})_{11} & V_{21}\Lambda_1 (V^{-1})_{12} \end{bmatrix} x(t) + \left(I_n - \begin{bmatrix} V_{12} (V^{-1})_{21} & V_{12} (V^{-1})_{22} \\ V_{22} (V^{-1})_{21} & V_{22} (V^{-1})_{22} \end{bmatrix} \right) B\bar{\bar{z}}$$

or

$$(47') \quad Dx(t) = \begin{bmatrix} V_{11}\Lambda_1 (V^{-1})_{11} & V_{11}\Lambda_1 (V^{-1})_{12} \\ V_{21}\Lambda_1 (V^{-1})_{11} & V_{21}\Lambda_1 (V^{-1})_{12} \end{bmatrix} x(t) + \begin{bmatrix} V_{11}(V^{-1})_{11} & V_{11}(V^{-1})_{12} \\ V_{21}(V^{-1})_{11} & V_{21}(V^{-1})_{12} \end{bmatrix} B\bar{\bar{z}}$$

$$t \geq t_f$$

Thus, on the stable manifold x(t) is indeed driven only by the stable roots.

Provided V_{11} and $(V^{-1})_{11}$ are of full rank, the state matrix in (47) and (47') has rank n_1. The behaviour of the predetermined variables ${}_1x(t)$ is given by:

$$D_1x(t) = V_{11}\Lambda_1(V^{-1})_{11}\,{}_1x(t) + V_{11}\Lambda_1(V^{-1})_{12}\,{}_2x(t)$$

$$+ \begin{bmatrix} V_{11}(V^{-1})_{11} & V_{11}(V^{-1})_{12} \end{bmatrix} B\bar{\bar{z}}$$

but from (46), for $t \geq t_f$

$${}_2x(t) = -[(V^{-1})_{22}]^{-1}(V^{-1})_{21}\,{}_1x(t) - [(V^{-1})_{22}]^{-1}\Lambda_2^{-1}[(V^{-1})_{21}(V^{-1})_{22}]\;B\bar{\bar{z}}$$

Therefore, for $t \geq t_f$

$$D_1x(t) = V_{11}\Lambda_1\left[(V^{-1})_{11} - (V^{-1})_{12}[(V^{-1})_{22}]^{-1}(V^{-1})_{21}\right]{}_1x(t)$$

$$+ V_{11}\{(V^{-1})_{11} - \Lambda_1(V^{-1})_{12}[(V^{-1})_{22}]^{-1}\Lambda_2^{-1}(V^{-1})_{21},\ (V^{-1})_{12}$$

$$- \Lambda_1(V^{-1})_{12}[(V^{-1})_{22}]^{-1}\Lambda_2^{-1}(V^{-1})_{22}\}B\bar{\bar{z}}$$

or

$$D_1x(t) = V_{11}\Lambda_1(V_{11})^{-1}\,{}_1x(t)$$

$$+ V_{11}\{(V^{-1})_{11} - \Lambda_1(V^{-1})_{12}[(V^{-1})_{22}]^{-1}\Lambda_2^{-1}(V^{-1})_{21},(V^{-1})_{12} - \Lambda_1(V^{-1})_{12}[(V^{-1})_{22}]^{-1}$$

$$\Lambda_2^{-1}(V^{-1})_{22}\}B\bar{\bar{z}}$$

$$D_1x(t) = -[(V^{-1})_{22}]^{-1}(V^{-1})_{21}\,D_1x(t)$$

For computational purposes this may well be superior to working with (1') directly, because it ensures that the inevitable numerical inaccuracies in the calculation of $x(t_f)$ will not put the system on a divergent trajectory.

Finally, if we analyse the behaviour of x in terms of deviations of x from its new steady state equilibrium $\bar{\bar{x}} = - A^{-1} B \bar{\bar{z}}$, i.e.

$$\tilde{x} \equiv \begin{bmatrix} {}_1\tilde{x} \\ {}_2\tilde{x} \end{bmatrix} \equiv x - \bar{\bar{x}}, \text{ we get for } t \geq t_f$$

$$D\,{}_1\tilde{x}(t) = V_{11}\, \Lambda_1 (V_{11})^{-1}\, {}_1\tilde{x}(t)$$

$${}_2\tilde{x}(t) = - [(V^{-1})_{22}]^{-1} (V^{-1})_{21}\, {}_1\tilde{x}(t) .$$

If the deviation of the output vector y from its new steady state equilibrium $\bar{\bar{y}} = [- CA^{-1} B + D]\, \bar{\bar{z}}$ is defined by $\tilde{y} = y - \bar{\bar{y}}$ then, for $t \geq t_f$

$\tilde{y} = C\tilde{x}$.

V The representation of the boundary conditions in rational expectations models as linear restrictions on the state vector: the number of stable eigenvalues exceeds the number of predetermined variables.

The issues addressed in this Section can be conveniently introduced with the help of the following example:

(48a) $m - p = ky - \lambda r \qquad k, \lambda > 0$

(48b) $y = \delta(e-p) \qquad \delta > 0$

(48c) $p = \alpha w + (1-\alpha)e \qquad 0 < \alpha < 1$

(48d) $Dw = \phi y + \Pi \qquad \phi > 0$

(48e) $De = r - r^*$

(48f) $D\Pi = \beta(Dp-\Pi) \qquad \beta > 0$

(48g) $\ell \equiv m - w$

(48h) $c \equiv e - w$

Equations (48 a-f) describe a small open economy with perfect international capital mobility, perfect substitutability between domestic and foreign bonds and risk neutrality. The notation is as follows: m is the nominal money stock, p the domestic price level, y real output, r the domestic nominal interest rate, e the exchange rate (number of units of domestic currency per unit of foreign currency), w the money wage, Π the underlying or "core" rate of inflation, r^* the world interest rate. Equation (48a) is the LM curve. Equation (48b) describes the IS curve. Equation (48c) says that the price of domestic output is a mark-up on unit labour cost and unit import costs. The augmented wage Phillips curve is in equation (48d). The international interest differential equals the expected rate of exchange depreciation (equation 48e). The underlying or core rate of inflation adjusts adaptatively to the excess of the actual rate of inflation over the underlying rate of inflation (equation 48f). All variables except r, r^* and Π are in logarithms.

A minimal representation of the dynamic system involves 3 state variables. We choose Π, ℓ (real money balances) and a c (competitiveness). The state-space representation, using the format of equations (1') and (2') is given in (48a, b).

$$(49a)\quad \begin{bmatrix} D\ell \\ Dc \\ D\Pi \end{bmatrix} = \begin{bmatrix} 0 & -\psi\alpha\delta & -1 \\ -\lambda^{-1} & -\lambda^{-1}(\alpha\delta(\phi\lambda-k)+\alpha-1) & -1 \\ -\beta(1-\alpha)\lambda^{-1} & \beta[\alpha^2\phi\delta+\lambda^{-1}(1-\alpha)(1-\alpha(1-\delta k))] & -\beta(1-\alpha) \end{bmatrix} \begin{bmatrix} \ell \\ c \\ \Pi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -\beta(1-\alpha) \end{bmatrix} \begin{bmatrix} Dm \\ r^* \end{bmatrix}$$

$$(49b)\quad \begin{bmatrix} r \\ y \\ Dw \\ Dp \\ De \end{bmatrix} = \begin{bmatrix} -\lambda^{-1} & \lambda^{-1}[k\alpha\delta+1-\alpha] & 0 \\ 0 & \alpha\delta & 0 \\ 0 & \alpha\phi\delta & 1 \\ -\lambda^{-1}(1-\alpha) & \alpha^2\phi\delta+\lambda^{-1}(1-\alpha)(1-\alpha(1-\delta k)) & \alpha \\ -\lambda^{-1} & \lambda^{-1}[k\alpha\delta+1-\alpha] & 0 \end{bmatrix} \begin{bmatrix} \ell \\ c \\ \Pi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -(1-\alpha) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Dm \\ r^* \end{bmatrix}$$

The money wage, w, is a predetermined or backward-looking variable. We also assume $m(t)$ to be a differentiable function of time. $\ell \equiv m - w$ is therefore a predetermined variable. Its boundary condition is

$$(50)\quad \ell(t_0) = \bar{m}(t_0) - \bar{w}(t_0) = \bar{\ell}(t_0)$$

The nominal exchange rate, e, is a forward-looking or jump variable. The real exchange rate, $e - w$ is therefore also a jump variable. The core rate of inflation, however, falls into neither category. First note that the determinant of the state matrix in (49a), i.e. the determinant of A in (1') is positive

$$|A| = \beta^{-1}\phi\delta\alpha > 0$$

Since $|A| = \prod_{i=1}^{n} \lambda_i$, this means that we either have three roots with positive real parts or one root with a positive real part and two roots with negative real parts. If there are three unstable roots, the model is completely unstable and nothing much can be said about it. However, for plausible values of the parameters, there will be two stable and one unstable root. Consider e.g. the parameter values $k = 1$, $\alpha = .75$, $\beta = .5$, $\lambda = 2$, $\delta = .5$ and $\phi = .5$. With these parameter values the trace of the state matrix in (49a) is zero. Since $\text{tr}\, A = \sum_{i=1}^{n} \lambda_i$, $|A| > 0$ and $\text{tr}\, A \lesseqgtr 0$ imply two stable roots and one unstable root if $n = 3$.[6/]

Since there are two stable and one unstable root, should Π be treated as predetermined? This would yield a partition of the state vector into two predetermined variables, ℓ and Π and one jump variable c. With the right number of stable and unstable roots, the methods of Section IV could be applied. It is clear, however, that Π is not a predetermined variable. From (48f), $D\Pi$ depends on Dp. From (48c) p is a function of e. e can make discontinuous jumps and so, therefore can p. Dp will not be defined at such points of discontinuity of e

and $\dot{p}$ (the left hand side derivative of $p(t)$ becomes infinite). Thus Π can jump in response to news but it will do so if and only if e (and therefore p and c) jumps. This suggests that the jump in Π will be a (linear) function of the jump in e (or in p or c). The following argument shows that this is indeed the case.

Equation (48f) can be rewritten as follows:

$$(51)\quad D(\Pi(t)\exp(\beta t)) = \beta Dp(t)\exp(\beta t)$$

or

$$\int_{t_o}^{T} D(\Pi(t)\exp(\beta t))dt = \beta\int_{t_o}^{T} Dp(t)\exp(\beta t)dt \qquad T > t_o$$

This can be written as

$$(52)\quad \Pi(T)\exp(\beta T)-\Pi(t_o)\exp(\beta t_o) = \beta\int_{t_o}^{T} Dp(t)\exp(\beta t)dt$$

Noting that $Dp(t) = \alpha Dw(t) + (1-\alpha)De(t)$

$$= (1-\alpha)Dc(t) - D\ell(t) + Dm(t)$$

we can rewrite (52) as

$$(53)\quad \Pi(T)\exp(\beta T) - \Pi(t_o)\exp(\beta t_o) = \beta(1-\alpha)\int_{t_o}^{T} Dc(t)\exp(\beta t)dt - \beta\int_{t_o}^{T} D\ell(t)\exp(\beta t)dt + \beta\int_{t_o}^{T} Dm(t)\exp(\beta t)dt$$

Integrating the terms on the right-hand side of (53) by parts we get

(54) $$\Pi(T)\exp(\beta T) - \Pi(t_o)\exp(\beta t_o) = \beta(1-\alpha)\left[c(T)\exp(\beta T) - c(t_o)\exp(\beta t_o) - \int_{t_o}^{T} c(t)\beta\exp(\beta t)dt\right]$$

$$\beta[\ell(T)\exp(\beta T) - \ell(t_o)\exp(\beta t_o) - \int_{t_o}^{T}\ell(t)\beta\exp(\beta t)dt]$$

$$+ \beta[m(T)\exp(\beta T) - m(t_o)\exp(\beta t_o) - \int_{t_o}^{T}m(t)\beta\exp(\beta t)dt]$$

Taking the limit as $T \to t_o$ (54) becomes

(55) $$\Pi(t_o^+) - \Pi(t_o) = \beta(1-\alpha)[c(t_o^+) - c(t_o)]$$

or

(55') $$\Pi(t_o^+) = \Pi(t_o) + \beta(1-\alpha)[c(t_o^+) - c(t_o)]$$

Where $x(t_o^+) \equiv \lim_{\substack{t\to t_o \\ t>t_o}} x(t)$.

To obtain (55) we have used $\exp(\beta t_o^+) = \exp(\beta t_o)$ and the assumption that $m(t)$ and $w(t)$ (and therefore $\ell(t)$) are predetermined, i.e. $\ell(t_o^+) = \ell(t_o)$ and $m(t_o^+) = m(t_o)$. (55) shows that the jump in Π at $t = t_o$, $\Pi(t_o^+) - \Pi(t_o)$ is a linear function of the jump in c, $c(t_o^+) - c(t_o)$. $\Pi(t)$ becomes a pure predetermined variable if the exchange rate has no direct effect on the price level, i.e. if $\alpha = 1$. Note that a boundary condition such as (55) fits into the general framework of (4) and (4'), as it can be rewritten as follows.

(56) $$1\,\Pi(t_o^+) - \beta(1-\alpha)c(t_o^+) + 0\,\ell(t_o) + 0\,\Pi(t_f) + 0c(t_f) + 0\ell(t_f) = \Pi(t_o) - \beta(1-\alpha)c(t_o)$$

The third boundary condition, in addition to (50) and (55) or (56) is the condition that a $t = t_f$ the system be on the 2-dimensional stable manifold. This will be a condition like (38). In the current example $\left(V^{-1}\right)_2$ is a (1x3) matrix and Λ_2 is a scalar.

Note, that although the boundary conditions involve c(t) at $t = t_o$ and at $t = t_f$, they can still be written as in (12a, b) because $c(t_o)$ and $c(t_f)$ do not occur in the same boundary condition. The analogue to (12a, b) is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\beta(1-\alpha) & 1 \end{bmatrix} \begin{bmatrix} \ell(t_o) \\ c(t_o^+) \\ \Pi(t_o^+) \end{bmatrix} = \begin{bmatrix} \bar{\ell}(t_o) \\ \Pi(t_o) - \beta(1-\alpha)c(t_o) \end{bmatrix}$$

and

$$\left(V^{-1}\right)_2 \begin{bmatrix} \ell(t_f) \\ c(t_f) \\ \Pi(t_f) \end{bmatrix} = -\lambda_2^{-1}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} V^{-1}B\begin{bmatrix} \bar{\bar{D}}m \\ \bar{\bar{r}}^* \end{bmatrix}$$

λ_2 is the unstable eigenvalue. $\left(V^{-1}\right)_2$, V^{-1} and B are as defined in Section IV. $\bar{\bar{D}}m$ and $\bar{\bar{r}}^*$ are the constant values of Dm(t) and r*(t) for $t \geq t_f$.

This can be generalized easily. An n-dimensional state vector could be partitioned into n_1 pure predetermined variables, n_2 pure jump variables or jump variables whose jumps are linearly independent and $n - n_1 - n_2$ jump variables whose jumps are linear combinations of the jumps in the pure jump variables, the "mixed" state variables. There no longer is equality between the number of predetermined variables and the number of stable eigenvalues. Now the number of stable eigenvalues must equal the number of predetermined and "mixed" state variables, while the number of unstable eigenvalues equals the number of pure jump variables.

The second and final example involves a system with two pure forward-looking or "jump" variables which nevertheless has only one unstable root.

$$\begin{bmatrix} Dx_1 \\ Dx_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} z$$

with $a_{11}a_{22} - a_{21}a_{12} < 0$

We can think of x_1 and x_2 as asset prices determined in efficient markets. Assume that for $t < t_o$ the system is in the steady-state equilibrium corresponding to the value of the forcing variable $z = \bar{z}$. At $t = t_o$ the news arrives that for $t \geq t_f \geq t_o$ z will assume the constant value $\bar{\bar{z}}$. The condition that the system converges to the steady-state corresponding to $\bar{\bar{z}}$ is clearly insufficient to determine a unique set of starting values for $x = (x_1, x_2)^T$ at t_o. Any jump of x that places it on the one-dimensional stable manifold corresponding to $\bar{\bar{z}}$ at $t = t_f$ will yield a convergent trajectory: there is a continuum of initial values for x consistent with asymptotic convergence.

If however, the not implausible boundary condition is imposed that at $t = t_f$ the system be at the new long run equilibrium, i.e. $\begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = \bar{\bar{x}} =$

$$= - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \bar{\bar{z}}$$

, a unique starting value for x at $t = t_o$ will be determined. That starting value is the only one that places the system at the new steady state as soon as the forcing variables assume their new stationary values. Since both x_1 and x_2 are forward-looking jump variables there would seem to be no good reason for making them "prisoners of the past". Whatever the economic merits of the case, the boundary conditions just given fit the general structure of the two-point boundary value problem solution methods described in Section III.

The examples bring out a quite general feature of the saddle point problems that frequently arise in the solution of rational expectations models. If the number of unstable characteristic roots exceeds the number of jump variables (or equivalently, if the number of stable characteristic roots is less than the number of predetermined variables) no convergent solution trajectory will in general exist. It is, however, possible for the number of unstable characteristic roots to fall short of the number of jump variables (or equivalently, for the number of stable characteristic roots to exceed the number of predetermined variables), without this necessarily implying that there exists an infinite number of convergent solutions. A unique convergent solution can still exist provided the boundary conditions on the n state variables yield n linearly independent equations in (10) or (10a). In terms of equations (12a, b), a unique solution requires that the rank of the matrix $\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ be n.

Conclusion

A class of linear two-point boundary value problems has been analysed which has many applications in macroeconomics. The boundary conditions have been expressed as linear restrictions on the state vector at an initial time t_o and at a finite terminal time t_f. This goes through even if the terminal conditions involve the (asymptotic) convergence of the system to a steady state equilibrium. A generalization is given of the condition that the number of stable eigenvalues equals the number of predetermined or backward-looking state variables and that the number of unstable eigenvalues equals the number of forward-looking or jump state variables.

Footnotes

1/ I am abstracting from two classes of non-uniqueness problems that arise even in linear rational expectations models. These are (1) the choice between backward or forward solutions or convex combinations of the two and (2) the problem of extraneous noise ("sunspots") entering the solution in addition to "market fundamentals". See Blanchard [1979], Flood and Garber [1980], Buiter [1980].

2/ $z(t)$ is a right-continuous function of time, i.e. $\lim_{\substack{t \to t' \\ t > t'}} z(t) = z(t')$.

It is also assumed to be bounded and to have only a finite number of points of discontinuity on any closed interval. $z(t)$ is therefore an integrable function of time.

3/ Note that (14a) is a nonlinear matrix differential equation. It can always be rearranged into a vector differential equation. Let Q be an $n \times m$ matrix. vec (Q) is the nm-element column vector whose first n elements are the first column of Q; the second n elements are the second column of Q etc. Then (14a) can be rewritten as:

$$D \text{ Vec}\left(S(t)\right) = \left(S^T(t) \otimes I\right) \text{Vec}\left(F_1\right) - \left(F_4^T \otimes I\right) \text{Vec}\left(S(t)\right)$$

$$- \left(I \otimes S(t) F_3\right) \text{Vec}\left(S(t)\right) + \text{Vec}\left(F_2\right)$$

Alternatively, it can be noted that (14a) is a Riccati matrix differential equation for which known solution methods exist (Reid [1972]).

4/ Sometimes the weaker conditon is imposed that if the values of the forcing variables $z(t)$ are bounded, then the values of the state variables should remain bounded. For practical purposes it would seem that little generality is lost by assuming, that after some point in time $t_f < \infty$, the forcing variables remain constant.

5/ The essential simplifying assumption we make is that A is diagonalizable. n distinct eigenvalues for A is a sufficient condition for diagonalizability. Even if A has eigenvalues with multiplicity greater than one, it is diagonalizable as long as A has n linearly independent eigenvectors. This will happen if and only if there are k linearly independent eigenvectors corresponding to each eigenvalue of multiplicity k, that is i.f.f. for each value λ of A the multiplicity of λ equals the nullity of $\lambda I - A$.

6/ A slightly less open economy, $\alpha = .8$, with a higher speed of adjustment of Π, $\beta = \frac{2}{3}$, yields, tr $A = -.0333333$ ..

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