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Multiple Shooting in Rational Expectations Models

ABSTRACT

This note describes an algorithm for the solution of rational expectations models with saddlepoint stability properties. The algorithm is based on the method of multiple shooting, which is widely used to solve mathematically similar problems in the physical sciences. Potential applications to economics include models of capital accumulation and valuation, money and growth, exchange rate determination, and macroeconomic activity. In general, whenever an asset price incorporates information about the future path of key variables, solution algorithms of the type we consider are applicable.

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The assumption of rational expectations or perfect foresight has been found useful in many fields of economics. While rational expectations models have received extensive theoretical attention to date, there have been relatively few applications.[1] In part, this has been due to the difficulty of numerically calculating the saddlepoint paths, which typically represent the solution of these models. Blanchard and Kahn (1980) present an algorithm for the solution of linear rational expectations models. In this note we extend their analysis by presenting an algorithm for the solution of general non-linear rational expectations models.

Formally, rational expectations models pose two-point boundary value problems, which are common in optimal control, engineering, and the physical sciences. When the system under study is linear, an analytic solution for the two-point boundary value problem can be found. For non-linear systems, this is not possible and numerical techniques are almost always necessary. This note shows how the method of multiple shooting, utilized in the physical sciences, can be used to solve a wide variety of economic models. The first section describes some of the economic contexts in which two-point boundary value problems arise. The method of multiple shooting and its application to economic models are examined in the second section. There is a brief conclusion.

I. Economic Application of Two-Point Boundary Value Problems

The class of two-point boundary value problems considered here has the form:

$$x_{t+1} = F(x_t, z_t, t) \qquad (1)$$

where

x_t is n x 1

z_t is k x 1

$F: \mathbb{R}^{k+n} \rightarrow \mathbb{R}^n$

$x_t = [v_t, w_t]$

$v_t = \bar{v}$ v is m x 1

$w_0 = \bar{w}$ w is (n-m) x 1 (2)

The dynamic economic system is defined by (1) and (2). Equation (2) defines boundary conditions at the beginning and end of the interval of interest.[2] Because the problem involves a mixture of initial and terminal conditions it is referred to as a two-point boundary value problem. The exogenous variables entering the model are represented by z_t, with the entire sequence $\{z_i\}$ $i = 0, T$ given. We allow for the possibility that F may be time dependent, (i.e., non-autonomous). In many economic problems the time horizon is infinite. The boundary value conditions are not given by $v_T = \bar{v}$ but rather $\lim_{t\to\infty} v_t = \bar{v}$. We assume throughout that the latter case can be adequately approximated by the former for sufficiently large T.[3]

Models of the type illustrated by (1) and (2) have widespread application in economics. A simple example is provided by a standard model of capital accummulation.[4] Arbitrage requires that the yield on capital equal the return on the alternative asset. That is:

$$\frac{f'(K)}{p_K} + \frac{\dot{p}_K}{p_K} = r \qquad (3a)$$

or

$$\dot{p}_K = rp_K - f'(K) \qquad (3b)$$

where $f'(K)$ is the marginal product of capital, and p_K is its price. The supply of new capital goods depends on their price p_K, according to the investment schedule

$$\dot{K} = I(p_K) \qquad I'>0 \quad I(1) = 0 \qquad (4)$$

where the normalization $I(1) = 0$ is chosen for convenience. The path of capital accummulation and valuation may be found by solving the pair of differential equations (3b) and (4). The boundary conditions are provided by the initial capital stock K_0 and the requirement that the model converge to a steady state so that $\lim_{t\to\infty} p_K = 1$. Since boundary conditions are imposed at both the beginning and end of the solution interval, the model fits the form of the canonical two-point value problem (1) and (2). This example is particularly simple since there are no exogenous variables and the behavioral functions are non-autonomous.

In addition to models of capital accummulation and valuation, a wide range of economic models from many fields present two-point boundary value problems. Whenever an asset price or shadow price incorporates information about the future path of key variables, solution algorithms of the type we consider are applicable. Any intertemporal optimization problem with constraints falls within the two-point boundary value class. Solving such problems requires initial conditions on the state variables, and terminal conditions on the costate variables corresponding to the constraints. Other

applications include models of money and growth of the type considered by Sargent and Wallace (1973), exchange rate determination as discussed in Dornbusch (1976), and macro-economic activity of the type described in Blanchard (1978) and Fair (1979).

Previous attempts to solve rational expectations models which pose two-point boundary value problems include Blanchard and Kahn (1980) and Fair (1979). Blanchard and Kahn present an explicit analytical solution for linear models, which is implemented computationally in Blanchard (1979). Fair uses a numerical version of his econometric model. His method may be difficult to apply to problems with more than a very small number of terminal conditions.

II. The following is an informal summary of the method of multiple shooting. Detailed presentations of multiple shooting may be found in Roberts and Shipman (1972) and Keller (1968). As Roberts and Shipman indicate, all shooting algorithms for two-point boundary value problems have the following basic structure:

> (A) set of values of the unspecified conditions at the initial point of the interval ("missing initial conditions") is assumed, and the differential equations are numerically integrated to the terminal point ("shooting" at the target terminal points). If the computed terminal values satisfy the specified terminal conditions the problem has been solved. If they do not (the normal course of events), the differences between the computed and specified terminal conditions (the "miss distances") are used to adjust the missing initial conditions. If the differential equations and boundary conditions are linear, the adjustment need only be made once, but if the differential equations or the boundary conditions are nonlinear, the adjustment of the missing initial conditions is an iterative procedure.
> (p. xiii)

In the standard shooting method, the system is integrated all the way from the initial period to the terminal period. In explosive, saddlepoint-stable systems, this integration or other aspects of the shooting procedure may break down. Multiple shooting provides a more robust procedure for these difficult cases. In multiple shooting, the interval of interest is subdivided, auxilliary variables are defined and auxilliary intermediate-point conditions are imposed. Guesses of initial conditions are made not just for the beginning period, but for many intermediate periods as well. By breaking up the shooting problem into stages, the explosive character of saddle-point systems may be numerically brought under control.

Shooting methods apply to both differential and difference equation systems. Indeed the numerical solutions for the former are typically found after discretization of the system. We will summarize here the procedure for difference equation models.

By repeated function composition, we may integrate (1) forward to write:

$$x_T = H[x_0;\ \{z_i\}_{i=0,T}] \tag{5}$$

Our goal is to find a v^* such that:

$$x_T = H([v_0^*,\ \bar{w}]\ ;\ \{z_i\}_{i=0,T}) = [\bar{v},\ w_T]\ . \tag{6}$$

Corresponding to each root v_0^* of this equation is a solution of the two-point boundary value problem.[5] Thus the solution of the differential equation system (1) and (2) can be reduced to the solution of the set of non-linear equations specified in (6). A standard approach to finding the solutions of (6) is the application of Newton's method as described below.

Consider the function H^m mapping domain of H into the first m elements of its range. Then, $v_t = H^m[x_0 ; \{z_i\}] = H^m[v_0, \bar{w} ; \{z_i\}]$. We write H^m in this way to stress the functional dependence of v_T on v_0 . For given $\bar{w}$ and $\{z_i\}$ H^m maps $\mathbb{R}^m \to \mathbb{R}^m$, from v_0 to v_T . By a first-order Taylor approximation, we have:

$$v_T(v_0) = H^m[v_0] \simeq H^m[\hat{v}_0] + \frac{\partial H^m}{\partial v_0}(v_0 - \hat{v}_0) \qquad (7)$$

where $\hat{v}_0$ is the point around which H^m is linearized. (Thus, $\frac{\partial H^m}{\partial v_0}$ is evaluated at v_0 .) (Note that $\bar{w}$ and z_i have been suppressed in the notation.) By definition, $v_t(v_0^*) = \bar{v}$. Using (7), we have:

$v_T(v_0^*) - v_T(\hat{v}_0) \simeq \frac{\partial H^m}{\partial v_0}(v_0^* - \hat{v}_0)$. Thus assuming $\left[\frac{\partial H^m}{\partial v_0}\right]^{-1}$ exists,

$$v_0^* \simeq \hat{v}_0 - \left[\frac{\partial H^m}{\partial v_0}\right]^{-1} [v_T(\hat{v}_0) - v_T(v_0^*)] \qquad (8)$$

If H is linear, the Taylor expansion is exact, and (8) is a strict equality. If not, the equation suggests an iterative procedure. Starting with a guess v_0^1 for v_0^* , we compute for all i,

$$v_0^{i+1} = v_0^i - \left(\frac{\partial H^m}{\partial v_0^i}\right)^{-1} [\bar{v} - v_T(v_0^i)] \qquad (9)$$

Here $\frac{\partial H^m}{\partial v_0}$ is evaluated each iteration at $[v_0^i, \bar{w}, y_0, \{z_i\}_{i=0,T}]$.[6]

Conditions on H for the convergence of this process may be found in Roberts and Shipman (Chapter 6).

Multiple Shooting

The simple Newton search typically fails for dynamic systems with saddlepoint stability. Incorrect guesses of v_0 are magnified through time, so that huge errors are recorded in v_T. $\frac{\partial H}{\partial v_0}$ may become ill-conditioned, or the first-order Taylor approximation may be so poor as to cause the iterations to move in the wrong direction. Also, it may become impossible to solve the F function at some intermediate step, before T is reached. A better-behaved algorithm is established by breaking the required search into stages. The algorithm purchases improved stability at the cost of increasing the size of the system of nonlinear equations which must be solved. Now divide the interval $[0,T]$ into intervals: $[0,T_1]$, $[T_1,T_2]$, ..., $[T_{N-1},T_N]$, with $T_N = T$. Using (1), the following implicit functions may be characterized:

$$x_{T_1}(v_0) = H_1(v_0;\ \bar{w}_0,\ J([v_0,\ w_0]),\ \{z_i\}_{i=0,T_1}) \qquad (10)$$

$$x_{T_2}(x_1) = H_2(x_1;\ J(x_1,\{z_i\}),\ \{z_i\})$$

$$\vdots$$

$$v_{T_N}(x_{N-1}) = H_N(x_{N-1};\ J(x_{N-1},\ \{z_i\}),\ \{z_i\})$$

The notation in (10) is important. The x_i, $i=1,\ldots,N$ are values of the x vector at the beginning of interval $(i + 1)$. Thus, x_1

is the starting vector of interval $[T_1, T_2]$. The $x_{T_i}(x_{i-1})$ are the final values of x in interval i , written as implicit functions of the x_{i-1} . The dimensions of the H_i differ: H_1 maps $\mathbb{R}^m \to \mathbb{R}^n$, $1<i<N$, H_i maps $\mathbb{R}^n \to \mathbb{R}^n$, and H_N maps $\mathbb{R}^n \to \mathbb{R}^m$. Defining the stacked (column) vectors $\underset{\sim}{X} = [v_0, x_1, x_2, \ldots, x_N]$, and $\tilde{\underset{\sim}{X}} = [x_{T_1}, x_{T_2}, \ldots, v_{T_N}]$, (10) may be concisely written as:

$$\tilde{\underset{\sim}{X}} = \underset{\sim}{H}(\underset{\sim}{X}) \quad \text{with} \quad \underset{\sim}{H} : \mathbb{R}^{(Nn+m)} \to \mathbb{R}^{(Nn+m)} \tag{11}$$

It is convenient to define $\tilde{\tilde{\underset{\sim}{X}}} = [x_1, x_2, \ldots, \bar{v}]$, and

$$\tilde{\underset{\sim}{M}} = \tilde{\underset{\sim}{X}} - \tilde{\tilde{\underset{\sim}{X}}} = [x_{T_1} - x_1, x_{T_2} - x_2, x_{T_3} - x_3, \ldots, v_{T_N} - \bar{v}] \ .$$

Since $\tilde{\underset{\sim}{X}}$ is a function of $\underset{\sim}{X}$, $\tilde{\underset{\sim}{M}}$ is implicitly a function of $\underset{\sim}{X}$. In particular, if there exists an $\underset{\sim}{X}^* \mid \tilde{\underset{\sim}{M}}(\underset{\sim}{X}^*) = 0$, then v_0^* , the leading terms of $\underset{\sim}{X}^*$ will satisfy the terminal conditions in the differential equation system (1). Thus, our goal is to locate $\underset{\sim}{X}^*$. Again, we turn to Newton search. With $\tilde{\underset{\sim}{M}} = \tilde{\underset{\sim}{M}}(\underset{\sim}{X})$,

$$\tilde{\underset{\sim}{M}}(\underset{\sim}{X}) \simeq \tilde{\underset{\sim}{M}}(\hat{\underset{\sim}{X}}) + \frac{\partial \tilde{\underset{\sim}{M}}}{\partial \underset{\sim}{X}} [\underset{\sim}{X} - \hat{\underset{\sim}{X}}] \tag{12}$$

Since $\tilde{\underset{\sim}{M}}(\underset{\sim}{X}^*) = 0$,

$$\underset{\sim}{X}^* \simeq \hat{\underset{\sim}{X}} - \left[\frac{\partial \tilde{\underset{\sim}{M}}}{\partial \underset{\sim}{X}}\right] [\tilde{\underset{\sim}{M}}(\hat{\underset{\sim}{X}})] \tag{13}$$

This relation suggests the algorithm for updating guesses:

$$\underset{\sim}{X}^{i+1} \simeq \underset{\sim}{X}^{i} - \left[\frac{\partial \tilde{\underset{\sim}{M}}}{\partial \underset{\sim}{X}}\right] [\tilde{\underset{\sim}{M}}(\hat{\underset{\sim}{X}}^{i})] \tag{14}$$

It is instructive to examine the Jacobian a bit more carefully. When fully written out in terms of the underlying H functions we find:

$$\frac{\partial \tilde{M}}{\partial \underset{\sim}{X}} = \begin{bmatrix} \left[\frac{\partial H_1}{\partial v_0}\right]_{nxm} - I_{nxm} & [0]_{nx(N-1)n} \\ [0]_{nxm} & \left[\frac{\partial H_2}{\partial x_1}\right]_{nxm} - I_{nxm} \quad [0]_{nx(N-3)n} \\ \vdots & \vdots \\ [0]_{nx(i-1)n} & \cdots \left[\frac{\partial H_i}{\partial x_{i-1}}\right]_{nxn} - I_{nxn} \quad [0]_{nx(n-3)n} \\ \vdots & \vdots \\ [0]_{mx(N-1)n} & \cdots \left[\frac{\partial H_N}{\partial x_{N-1}}\right]_{mxn} \end{bmatrix} \qquad (15)$$

Notice that $\frac{\partial \tilde{M}}{\partial \underset{\sim}{X}}$ has a band structure, which greatly facilitates the necessary matrix inversion.

For many problems which arise in practice, it is not possible to specify explicitly equations of motion of the form (1). Often the dynamics of the system are defined implicitly by a system of equations involving other endogenous variables. That is:

$$x_{t+1} = F(x_t, y_t, z_t, t) \tag{16a}$$

$$0 = G(x_t, y_t, z_t, t) \tag{16b}$$

y_t is $j \times 1$

$F : \mathbb{R}^{n+j+K} \to \mathbb{R}^n$

$G : \mathbb{R}^{n+j+K} \to \mathbb{R}^j$

where y_t and the range of G are of equal dimension. The implicit function theorem insures that under certain regularity conditions; (G everywhere differentiable with non-singular Jacobian), (16) is equivalent to (1). In practice it may not be possible however to solve (16). In this case, it is necessary to use an equation solving routine to solve (16) at each integration step.

III. Conclusion

The approach we have described above has proved useful in a wide variety of numerical problems.[7] In view of the important position which rational expectations models occupy in current economic research, we anticipate that this approach shall find application in many fields which we have not discussed. The authors welcome comparisons of multiple shooting with alternative solution approaches.

FOOTNOTES

1. Prominent exceptions include Fair (1979) and Blanchard (1979).

2. The methods discussed can also handle far more general boundary conditions. Any set of n linearly independent restrictions on the $\{ x_i \}_{i=0,T}$ is acceptable.

3. This approximation is made quite often in solving two point boundary value problems. Robertson (1971) discusses some approaches for testing the sensitivity of solutions to changes in T.

4. This model may be traced back as far as Walras. It is the essence of two-sector theories of investment and plays a key part in "Keynesian" investment models based on internal adjustment costs. A full discussion of such models may be found in Hayashi (1979).

5. A rigorous proof of this proposition along with a discussion of necessary and sufficient conditions for a unique v_0 to exist may be found in Keller (1968).

6. In the "modified" Newton's method, $\frac{\partial H^m}{\partial v_0}$ is evaluated at one point, v_0^1. While computationally easier, this modification may substantially slow the rate of convergence.

7. A detailed description of the algorithm is available on request from the authors.

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