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A LIFE CYCLE MODEL OF THE HOUSEHOLD'S TIME ALLOCATION

By Malcolm S. Cohen and Frank P. Stafford*

A model is developed to explain simultaneously the number of children born, the level of expenditures on children and other expenditures, the spacing of children and labor force participation and time devoted to child care by the husband and wife. The model is a control theory problem. It is solved by a computer simulation which illustrates how a husband-wife family would behave if it attempted to optimize the sum of lifetime utility. The family's utility is assumed to depend on the level and time path of consumption, number of children and time spent on leisure and child care.

1. Introduction

In this paper we present some preliminary results of research on a life cycle model which attempts a rather ambitious integration of several important aspects of household behavior. Most research on household behavior treats each aspect of choice to be largely independent of other choices. Studies of consumption behavior have assumed wages, family composition, and other factors as given and have considered the optimal intertemporal allocation of market expenditures (Modigliani and Brumberg, 1954). The life cycle of investment in training has received considerable attention but has focused on a simple choice between training and earning (Ben–Porath, 1970). Demographic studies have considered the effect of income and other socioeconomic variables on family size and child spacing but have avoided or treated only imperfectly other aspects of life cycle behavior (Stafford, 1969; O'Hara, 1972). The literature on labor supply treats labor force participation as explained by exogenous variables such as family income, presence of children, wage of the individual and education (Cohen, Rea and Lerman, 1970). However, all the above mentioned variables are in fact simultaneously determined for any given household.

The recent theoretical work on fertility behavior (Willis, 1973; Becker and Lewis, 1973) has integrated labor supply decisions with the decisions on the quantity and quality of children through a comparative static framework. While this approach is illuminating, it is not well suited to portraying the time paths of the important decision variables. It is clear that major areas of household decision making are intertemporally dependent as well as interrelated with one another. Hence it could be fruitful to model the dynamic aspects of these jointly dependent decisions (spending and accumulating assets, training and supplying labor, and bearing and raising children). This paper represents our initial efforts in this direction and reports on some results from a large scale simulation model.

The model which we develop to integrate several aspects of household choice starts out as a two person model (husband and wife) and is intertemporal. The

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model is concerned with the allocation of time to the labor market, to consumption at home and to child care. The demographic parts of our model portray the birth (and spacing) of children and child care. The elements in our model other than children and their care are life cycle accumulation of human capital subsequent to formal schooling by the husband and wife, life cycle asset accumulation and consumption of market purchased goods, and allocation of time by the husband and wife over the life cycle.

In developing our current model we have resorted to explicit functional forms. The benefit of such an approach is that we can express our opinions about what we believe to be the nature of life cycle time and market goods allocation. The benefit of explicit functional forms is purchased at the price of our certainly being wrong in some of the particulars. However, our major goal is to emphasize several of the important kinds of intertemporal relations, and the economic problem with which we are attempting to deal is sufficiently complex that reliance on explicit functional forms is essential. Hence apart from the "correctness" of our particular model we are also arguing for a more comprehensive approach to household behavior. One obvious cost to such an intertemporal approach is that the dynamic model is sufficiently complex that closed form solution is virtually impossible. Some optimal control problems have closed form solutions (Athans and Falb, 1966). However, since this problem has no closed form solution we rely on simulation for an understanding of the model and this is presented in Section III while the model itself is presented in Section II. We conclude the paper with some overall comments in Section IV.

II. The Model

Our model is specified as a problem in discrete optimal control. The family has, at the initial time, values of various state variables: assets, children, potential earnings of the husband and of the wife. The family must choose a time path of various choice or control variables: expenditures on consumption and on children, a birth rate, labor supply of the husband and of the wife, and time spent on children and on leisure—so as to maximize a performance criterion (utility function). The analytic problem was simplified by putting terminal assets as a part of the performance criterion and by choosing a weight for terminal assets so as to result in an optimal plan having a terminal asset value near zero. Alternatively, we could have specified terminal assets (of zero, say) and treated the model as a two-point boundary value problem.

In this problem the performance criterion is

$$J = \sum_{t=0}^{T-1} (C_t^1 + C_t^2) + f(A_T) \tag{1}$$

where $t = 0$ is the given period of marriage and T is the given, last (retirement) period which is the end of the life span. Our model assumed the period of marriage to be fixed. However, Silver (1965) and Stafford (1969) considered variables affecting age of marriage. The $f(A_T)$ function declines sharply and approaches $-\infty$ as assets become increasingly negative. When A_T becomes large and positive $f(A_T)$ increases at a decreasing rate until $f(A_T)$ reaches an asymptote. For purposes of simulating the model the family's life was divided into ten five-year intervals; the family begins at age 20–24 and ends at age 65–69.

To introduce children and other consumption into the performance criterion we have an explicit disaggregation of consumption into C_t^1 and C_t^2:

C_t^1: Consumption other than that associated with children by the husband and wife.

C_t^2: Consumption associated with children by the husband and wife.

In our specification C_t^1 and C_t^2 can be viewed as more basic commodities in Becker's (1965) use of that term in that both time and market inputs must be combined to produce them. The allocation of time and money at each point in time is determined by the control variables $1_t^h, k_t^h, r_t^h, 1_t^w, k_t^w, r_t^w, X_{1t}, X_{2t}$ which are defined as follows:

1_t^h = Percent of time in the labor force by the husband

$$0 \leq 1_t^h \leq 1$$

k_t^h = Percent of time in the care of children by the husband.

$$0 \leq k_t^h \leq 1$$

r_t^h = Percent of time spent in producing (consuming) C_t^1 by the husband.

Since all time is used we have the equality condition on the husband's time that

$$1_t^h + k_t^h + r_t^h = 1. \tag{2}$$

Thus any one component of husband's time can be defined as a residual and only two control variables are necessary to define the husband's time allocation at a point in time. Similarly, the wife's time allocation in a period can be described by the variables $1_t^w, k_t^w, r_t^w$, defined in the same fashion as for the husband with

$$1_t^w + k_t^w + r_t^w = 1. \tag{3}$$

The amounts of market goods used per unit time producing C_t^1 and C_t^2 are priced at p_1, p_2 and are the control variables X_{1t} and X_{2t}, respectively. There is also the control variable for increasing the number of children in each period, n_t, with two restrictions on n_t. First, n_t is positive only for the childbearing age of the family (t = 0–5) and second, it is less than or equal to some biological limit b (like 3 per 5 year period). Hence, some families may choose to operate at a biological limit for part of their life. Further, children leave the household after some given time span (say 20 years).

Our control variables and their restrictions can be summarized as:

$$U = \begin{cases} X_{1t} \geq 0 \\ X_{2t} \gtreqless 0 \\ \left.\begin{matrix} 1_t^h \\ k_t^h \\ r_t^h \end{matrix}\right\} = 1 \\ \left.\begin{matrix} 1_t^w \\ k_t^w \\ r_t^w \end{matrix}\right\} = 1 \\ n_t \quad 0 \leq n_t \leq b \qquad 0 \leq t \leq T_1 \end{cases}$$

$$n_t = 0 \qquad T_1 < t \leq T$$

The state variables in our model are the rental value of the human capital or potential annual earnings of the husband (Y_t^h) and of the wife (Y_t^w), net assets (A_t) and the measures of the inventory of children (Z_T). With the potential earnings and the previously defined control variables, we can now enumerate the transition equation for net assets.

$$A_{t+1} - A_t = 1_t^h Y_t^h + 1_t^w Y_t^w - p_1 X_{1t} - p_2 X_{2t} - p_3 Z_t + p_4(A_t)A_t \tag{4}$$

where $p_4(A_t)$ is the yield on net assets and has a functional form of an exponential. The particular exponential relationship has sharply rising borrowing cost for net assets which are below some minimum balance and reaches an asymptote of the given market interest as net assets exceed the minimum. The asset equation also reflects the notion that there is some minimum per child expenditure on food and clothing (p_3) and this expenditure continues so long as the children are present in the household. The number of children present in the household is Z_t.

The potential wages of the husband and wife change according to the transition equations.

$$Y_{t+1}^h - Y_t^h = Y_t^h[\alpha_1(1_t^h)^2 + \alpha_2 1_t^h + \alpha_3 - \delta_2(t)] \tag{5}$$

$$Y_{t+1}^w - Y_t^w = Y_t^w[\alpha_4(1_t^w)^2 + \alpha_5 1_t^w + \alpha_6 - \delta_3(t)] \tag{6}$$

In (5) and (6) the parabolic relationship between percent of time in the labor force and increments to human capital allows for learning-by-doing as labor force participation increases. However, if α_1 or α_4 is negative and α_2 or α_4 positive, as we assume, too long a workday results in depreciation of human capital. The factors $\delta_2(t)$ and $\delta_3(t)$ are meant to represent time dependent (biological) depreciation of human capital. These depreciation rates ($\delta_2(t)$ and $\delta_3(t)$) should, in general, rise exponentially as the individual approaches retirement so as to decrease the potential wage.

A solution to this difference equation (5) or (6) is quite complicated. When solved as part of the system of equations which comprise our model, a closed form solution is probably impossible. Nevertheless a simplified partial equilibrium solution of (5) or (6) is instructive. Let $\delta(t)$ be $b_3 t^k$ for all t and then let us rewrite (5) as:

$$Y_{t+1}^h = Y_t^h[a + b^* + b_3 t^k] \quad \text{where:} \quad b^* = b_1 1_t^h + b_2(1_t^h)^2 \tag{5.1}$$

In the steady state where 1_t is constant:

$$Y_t = (a + b^*)^t Y_0 + b_3 \sum_{i=1}^{t} a^{t-1}(i)^k \tag{5.2}$$

If $(a + b^*) > 1$, Y_t is explosive for $b_3 \geq 0$.

However b_3 is negative so that $b_3 t^k$ can damp $a + b^*$ as the individual ages. Biological depreciation can overtake the returns from "learning by doing" on the job.

From our discussion of the control variable n_t we can define the number of children present and their average age by introducing a set of state variables,

Z_t^1, Z_t^2, Z_t^3 where the initial conditions on these are:

$$Z_t^1 = 0$$
$$Z_t^2 = 0$$
$$Z_t^3 = 0$$

and these equations of motion are given by

$$\left.\begin{array}{lll} (7.1) & Z_{t+1}^1 = n_t & \\ (7.2) & Z_{t+1}^2 = Z_t^1 & (= n_{t-1}) \\ (7.3) & Z_{t+1}^3 = Z_t^2 & (= n_{t-2}) \end{array}\right\} \begin{array}{l} Z_{t+1}^1 - Z_t^1 = n_t - Z_t^1 \\ Z_{t+1}^2 - Z_t^2 = Z_t^1 - Z_t^2 \\ Z_{t+1}^3 - Z_t^3 = Z_t^2 - Z_t^3 \end{array}$$

Number of children present is then given by:

$$(8) \qquad Z_t = n_t + \sum_{i=1}^{3} Z_t^i = n_t + n_{t-1} + n_{t-2} + n_{t-3}$$

The average age of the children which is a function of (7) is defined as follows (since each child born is present in the household for four five-year periods):[1]

$$(9) \qquad V_t = (n_t, Z_t^1, Z_t^2, Z_t^3)$$
$$= (Z_t)^{-1}(2\tfrac{1}{2}n_t + 7\tfrac{1}{2}Z_t^1 + 12\tfrac{1}{2}Z_t^2 + 17\tfrac{1}{2}Z_t^3) \qquad V_t = 0 \text{ for } Z_t = 0$$

In summary we have the state variables:

$$X = \begin{bmatrix} Y_t^h \\ Y_t^w \\ A_t \\ Z_t^1 \\ Z_t^2 \\ Z_t^3 \end{bmatrix} \text{ with initial conditions } \begin{bmatrix} \bar{Y}^h \\ \bar{Y}^w \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and no terminal conditions.}$$

At each point in time the husband and wife must decide how to allocate among various activities: market work, care of children, and consumption associated with children and with other goods. This implies that during each unit of time (say, one year) both the husband and wife have a unit time budget. The use of the time for home production has, at the margin, the opportunity cost of foregone marginal home production. This point is emphasized by Becker where he discusses rising marginal full prices of goods. In the time allocation approach of Becker, the opportunities facing the family at a point in time are defined not only by market opportunities but by the production relations for the more basic commodities. The consumption goods other than children are produced according to a single basic commodity in our model.

$$(10) \qquad C_t^1 = \gamma_0 \left[\beta_1 X_{1t}^{\beta_2} + (1 - \beta_1)\left(\frac{r_t^h r_t^w}{\beta_3}\right)^{-\beta_2} \right]^{-\gamma_1/\beta_2}$$ where γ_1 is less than 1 to reflect diminishing returns

[1] Our method of computing average age assumes the average age of all children born in a five year period is $2\frac{1}{2}$ years at the end of each five year interval.

The consumption associated with children is produced according to a somewhat more complex state of the art. There is an intermediate good, parental care (P_t) which is produced via the following CES relation:

$$P_t = \gamma_2[\beta_4 k_t^{h-\beta_5} + (1 - \beta_4)k_t^{w-\beta_5}]^{-1/\beta_5} \tag{11}$$

where the parameters have the usual CES restrictions (Ferguson, 1970). The size of β_5 determines the possibilities for substituting the one parent's time for the other's. In the limit ($\beta_5 = -1$) and the parents' time are perfect substitutes. It is our belief that substitution possibilities are, in fact, high and that relative wages tip the balance toward the parent with lower market wages. One could also argue for (or against) $\beta_4 \neq 0.5$; that the wife (or husband) is more effective in caring for children, and that this encourages her (him) to specialize in child care. This is not necessary as our simulations show. If the wife has a lower potential wage there will be incentive for a division of labor with the husband doing the earning and the wife caring for children. This results from the depreciation of human capital which occurs with low levels of labor force participation. A woman's liberation interpretation of this might be that the wife is discriminated against in attempting a labor force career but not in attempting a child care career.

Parental care is combined with expenditures on children to produce child care in another CES function:

$$M_t = \gamma_3(t)\left[\frac{(\beta_6 - V_t)}{\beta_6}P_t^{-\beta_7} + \frac{V_t}{\beta_6}(X_{2t})^{-\beta_7}\right]^{-1/\beta_7} \tag{12}$$

The usual CES parameter restrictions hold and a few comments about (12) are in order. The efficiency parameter γ_3 *decreases* over time to reflect biological limitations on the parents' ability to raise children as the parents age. The input intensity parameters ($1 - V_t/\beta_6$, V_t/β_6) depend on V_t (see equation 9) which is the average age of children. In this way the family's choice of a time path of n_t determines a time path of (technical) possibilities for combining own time and market goods in the provision of child care. A young child requires relatively more time while older children require relatively more market inputs. The extreme case of the college student represents the older child well; parents may not even see him and the only time required is that of mailing him a monthly support check. To complete the production relations for consumption related to children, C_t^2, we have

$$C_t^2 = \gamma_4[\beta_8 M_t^{-\beta_9} + (1 - \beta_8)Z_t^{-\beta_9}]^{-\gamma_5/\beta_9} \tag{13}$$

where $0 < \gamma_5 < 1$ to reflect diminishing returns to increased consumption per unit time of C_t^2 and β_9 should be chosen to reflect a low elasticity of substitution between care and numbers of children. The substitution between parental care and number of children relates to Becker's (1960) discussion of quality of children. It is clear that the extreme cases of fixed proportions and perfect substitution are not appealing. Fixed proportions would imply that all children receive equal care or that care per child is independent of the number of children the family has. Perfect (or "high") substitution would allow parents to reduce their total expenditures on children as the number of children present grew subsequent to marriage.

The elasticity of substitution in (13) and the age-of-children dependent intensity parameters in (12) are crucial in our model. They relate not only to the quality/quantity trade-off suggested by Becker but they also relate to the interpretation of the price of children across various socio-economic groups. While a high income family will find young children expensive in terms of the opportunity cost of time (and human capital losses), a high income family may have more children because when the children are older the parents can effectively substitute large amounts of market inputs for own time.[2] Moreover, if potential earnings of both husband and wife are high then the shadow price of raising younger children will be higher but the shadow price of older children will be lower.[3]

It should also be recalled that our model specifies some minimum maintenance cost per child that a family must spend. Such a cost is reflected in the asset equation (4) where p_3 reflects this minimum cost in each period that a child is present.

The effect of labor market participation on appreciation or depreciation of human capital of the husband and wife will operate to raise the price of more (or higher quality) young children for parents with high levels of human capital. Our model contains a highly simplified learning-by-doing human capital accumulation (depreciation) model, the period of early child care in which withdrawal from the labor market by husband or wife occurs puts an added cost to child care which is the depreciation of the husband's and/or wife's human capital. That withdrawal will be determined primarily by relative wages of husband and wife if the elasticity of substitution is high enough between husband's and wife's time in production of the intermediate child care good (P_t).

In our model, we have argued that there is a clear incentive for families to substitute more market inputs for their own time as the children grow older. This is consistent with everyday experience in which parents are observed to hire more fulltime baby sitters and nursery care as the child grows older. It is also consistent with the existence of the public school system. The argument that children are increasingly able to learn from their peers as well as their parents implies that activities organized using market inputs—nursery schools, kindergartens and the like are increasingly good substitutes for parents' own time.

From the point of view of any one family, the public school system is also an important input to the production of C_2. Although not a part of our current model, the impact of the school system on household behavior could be introduced in numerous ways, for example, by introducing a subsidy to the transition equation for assets. In reality, richer parents desire and obtain more "public" schooling for their children, but the essential goal of many publicly financed school systems with such features as state equalization is to provide a more nearly equal level of services to all school children independent of their parents' economic status. If such goals were attained, it is obvious that, according to our model, lower income parents residing in a highly subsidized public school system could easily have an incentive to "sell off" part of their subsidy. It is worth noting that a high elasticity of substi-

[2] An alternative (or possibly complementary) way of specifying the age of children dependency in producing M_t would be to allow the coefficient β_7, which defines the elasticity of substitution ($\sigma = 1/1 + \beta_7$), to be dependent on the average age of the children so that as children grow older it becomes easier to substitute market inputs, X_2, for parental care.

[3] The shadow price has two components: current forgone earnings associated with child care and loss of future earnings potential as specified by (5) and (6).

tution between the subsidy and other market inputs and between other market inputs and parents' own time makes it easier to provide the collective good in the presence of heterogeneous tastes and incomes of parents. If the school system is providing a subsidy below that desired by the parents, they can readily supplement that subsidy with their own time with none of the loss that would be implied by a low elasticity of substitution (in the extreme, fixed proportions). Further, the subsidy varies by grade school and high school, and we know that in most school systems, the level of resources per student increases substantially between grade school and high school. Johnson and Stafford (1973) report a ratio of grade school to high school expenditures of about 0.6. Hence, the transition from grade to high school will be marked by changed time allocation by the parents.

Given our model as outlined in equation (1) through (13) we can rewrite equation (1). The control variables operating through "production" relations define the performance criterion

$$\text{(1')} \qquad J = \sum_{t=0}^{T-1} (\gamma_0 \left[\beta_1 X_{1t}^{\beta_2} + (1 - \beta_1) \left(\frac{r_t^h r_t^w}{\beta_3} \right)^{\beta_2} \right]^{-\gamma_1/\beta_2} + \gamma_4 [\beta_8 M_t^{-\beta_9} + (1 - \beta_8) Z_t^{-\beta_9}]^{-\gamma_5/\beta_9}$$

where:

$$M_t = \gamma_3(t) \left[\frac{\beta_6 - V_t}{\beta_6} P_t^{-\beta_7} + \frac{V_t}{\beta_6} (X_{2t})^{-\beta_7} \right]^{-1/\beta_7}$$

and

$$P_t = \gamma_2 [\beta_4 (K_t^h)^{-\beta_5} + (1 - \beta_4)(K_t^w)^{-\beta_5}]^{-1/\beta_5}$$

or in general

$$\text{(1'')} \qquad J = \sum L_t(X, U) + \gamma A_T$$

In this problem our Hamiltonian is

$$\text{(14)} \qquad H_t = L_t(X, U) + \langle \lambda, f(X, U) \rangle$$

where $f(X, U)$ is the set of transition equations and λ is a set of adjoint variables and $\langle \; \rangle$ indicates inner product.

The necessary conditions for an optimal control path is that U be chosen so as to satisfy the equations of motion:

$$\text{(15)} \qquad X_{t+1} - X_t = \nabla_\lambda H_t(X_t, \lambda_{t+1}, U_t)$$

$$\text{(16)} \qquad \lambda_{t+1} - \lambda_t = -\nabla_x H_t(X_t, \lambda_{t+1}, U_t)$$

and that

$$\text{(17)} \qquad \nabla_u H_t(X_t, \lambda_{t+1}, U_t) + q_t' \frac{\partial_h}{\partial_{Ut}} = 0$$

where q_t is a vector of multipliers for our control variable constraints of the form

$$\text{(18)} \qquad h(U_t) \leq 0$$

to account for constraints on the control variables such as non-negativity of percent of time in that activity and that time add up to no more than 100 percent.

In (1′) C_t^1 and C_t^2 are additive at a point in time as well as over time. This may not be an ideal specification because there are presumably some substitution possibilities between C_t^1 and C_t^2 and the marginal contribution of C_t^1 is not independent of C_t^2 and conversely. However, our current specification does simplify the problem, allows for a positive utility even if the family has no children, and allows substitution between children and other consumption via the production relations. Further, since both elements of consumption are subject to diminishing returns per unit time there is incentive to smooth out consumption over time.

III. Simulation

The goal of our simulation is primarily to establish that a life cycle model of the type we have developed can be used to portray some of the qualitative behavior of a "typical" household. Any attempt to use the model to characterize numerous subgroups of the population would be a more difficult task since one would have to make guesses about the joint distribution of critical parameters across various social groups. However, simulation of such a model is suggestive of a group with parameters corresponding to those chosen *a priori*. There are two potential benefits from work with such a model. First, simulation illustrates interdependencies in household choice. For example, a higher rate of interest influences a wide range of household behavior ranging from lowered borrowing early in life, greater labor force participation, fewer children and greater life cycle savings. Second, one can gain an appreciation for the enormous complexity of the dynamics of a system of relations which might be viewed as a plausible representation of the household. Social scientists often make intuitive statements about household behavior which if pursued deliberately would produce a formal model with properties surely as difficult to appreciate fully as the one we have portrayed. Child spacing is an example of this point because a family's time path of a birth rate if put in the context of an optimizing model will invariably require some specification of time allocation of the husband and wife which in turn influences earnings potential of each.

A conjugate gradient algorithm was used to simulate the control path which would minimize the negative of the objective function (J) in equation (1) with the search for the minimum terminated when an additional iteration increased (j) by less than 0.001 or approximately 0.0033 percent. The conjugate gradient algorithm is discussed by Reeves (1964) and Bryson and Ho (1969). Chosen values of the parameters were set based on *a priori* assumptions about the relative importance of the two types of consumption, the degree of substitutability between various control variables and in order to accomplish certain scaling. While parameters of the model depend on certain observables such as the interest rate, other parameters are quite difficult to estimate empirically such as the elasticity of substitution between a husband and a wife's time in the provision of child care and reflect best guesses. Once the parameters are set the algorithm will pick a set of 53 control variables which minimize $-J$. In practice it is not possible to pick the 25 parameters in the model and expect a reasonable time path for the control variables since it is first necessary to scale the parameters in some consistent way. Therefore a number of runs were necessary to accomplish this scaling. Once the scaling was established it was possible to try alternative parameter changes to test the sensitivity of certain parameters and assumptions.

Two problems arose in simulating the optimal time paths which are of general interest.

1) Attempts were made to reduce the 53 control variables to be simulated making restrictive assumptions about the time paths of the control variables.

2) Attempts were made to restrict the control variables from taking on unreasonable values.

The first problem was handled by assuming the control variables followed a time path approximated by a Kth degree polynomial. Thus a control variable could be approximated by:

$$C_t = b_0 + b_1 t + b_2 t^2 + \ldots b_K t^K \qquad t = 0, 9 \tag{19}$$

Instead of having to estimate 53 control variables, we found it possible to get a good fit only estimating 28 variables. The seven control variables were approximated by a polynomial of the third degree. The degree of the approximating polynomial chosen depends on our *a priori* assumptions about the time path of the seven control variables. If we assume a linear time path a polynomial of the first degree is adequate. Because we did not want to prejudge the degree of the approximating polynomial the simulations shown in this paper were run for all 53 control variables.

To restrict the control variables from taking unreasonable values we built internal constraints into the model as well as using a penalty function. The internal constraints consisted of defining two time variables for the husband and two for the wife. The two variables were percent of time spent on non-work and percent of non-work time spent on child care. The two variables permit us to split time into three parts without worrying about total time for the three activities exceeding 100 percent, unless either variable is less than 0 percent or more than 100 percent.

To reduce the likelihood that an optimal time path will result with negative time or time above 100 percent or negative money spent we set time spent to 0 or 100 percent when an iteration reached a non-permissible boundary and set negative money spent to zero. This procedure per se resulted in some problems in computing gradients. If an iteration started with a large negative percent of time the algorithm would not iterate away from 0 to a positive value. To solve this problem we introduced a penalty function such that:

$$F_t = f U_t^2 \qquad U_t < 0 \tag{20}$$

where f is a penalty and U_t is a control variable and

$$F_t = g(U_t - 1)U_t \qquad U_t > 1 \tag{21}$$

and

$$F_t = h(n_t - b)^2 \qquad n_t > b \tag{22}$$

where n_t is the number of children born and b is the biological limit.

Table 1 summarizes the values of the control variables and state variables in a converged simulation. Some of the notable results include:

TABLE 1

AN ILLUSTRATIVE SIMULATION

Control Variables	20–24	25–29	30–34	35–39	40–44	45–49	50–54	55–59	60–64	65–69
Consumption (thou $/yr.) ($X_1$)	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0
Leisure *Head* (% of time) (R_H)	26.0	23.0	23.0	22.0	29.0	42.0	49.0	72.0	99.0	100.0
Child Care *Head* (% of time)	26.0	29.0	28.0	29.0	24.0	8.0	0.0[a]	0.0[a]	0.0[a]	0.0[a]
Work *Head* (% of time)	48.0	48.0	49.0	49.0	47.0	50.0	51.0	28.0	1.0	0.0
Leisure Wife (% of time)	28.0	25.0	24.0	25.0	33.0	49.0	65.0	92.0	100.0	100.0
Child Care *Wife* (% of time)	29.0	34.0	36.0	41.0	38.0	17.0	0.0[a]	0.0[a]	0.0[a]	0.0[a]
Work *Wife* (% of time)	43.0	41.0	40.0	34.0	29.0	34.0	35.0	8.0	0.0	0.0
Consumption spent on children (in thou $/yr.)										
Extra	0.4	0.6	0.8	0.9	1.0	1.1	1.1	0.0[a]	0.0[a]	0.0[a]
Children born	0.8	0.5	0.5	1.0	0.0[a]	0.0[a]	0.0[a]	0.0[a]	0.0[a]	0.0[a]
Other Variables										
Average Age of Children	2.0	5.1	7.8	8.8	10.6	13.6	17.0	0.0	0.0	0.0
Number of Children Present	0.8	1.3	1.8	2.8	2.0	1.5	1.0	0.0	0.0	0.0
Income of Husband[b]	9.9	11.9	14.2	16.6	18.9	20.3	19.6	15.6	8.6	0.1
Income of Wife[b]	6.0	7.2	8.7	10.3	11.8	12.6	12.2	9.9	4.2	0.0
Assets at end of period (thou/$)	−3.6[c]	−6.4	−6.6	−12.4	−6.3	18.2	52.4	57.4	33.5	4.7

[a] Set to zero by assumption.

[b] Potential income in thou/$ per year. $J = -31.23$.

[c] Assets at beginning of period = $0.0. Fixed cost per child $2,000/year.

1. The family chooses a level consumption X, of \$6,000/year, (where X_1 is consumption not used for the care of children). On the other hand, X_2 rises over time (consumption spent on children) generating an inverse *u*-shape to total market expenditure $(X_1 + X_2)$.

Level consumption on X_1 is consistent with assumed diminishing returns per unit time to consumption of type 1 encouraging the spreading out of consumption. Since a positive rate of interest was assumed, consumption later in life would be less costly than consumption earlier in life, *ceterius paribus*. However, our model suggests that time is also more costly earlier in life. The family would therefore demand more goods intensive consumption earlier in life rather than time intensive consumption. These two effects could easily neutralize one another and this happens for the particular simulation in Table 1. This particular simulation resulted in a flat level of X_1 because the degree of the C_1 production function was very low assuring very sharp diminishing returns to consumption per unit of time. Thus the positive interest rate was not sufficient to induce the family to bunch consumption.[4]

2. Consumption spent on children is of two types. Each child born is assumed to cost a fixed sum of \$2,000 per year while the child is at home ($V \leq 20$). The simulation affects this total cost only by determining the number of children born. The family also obtains utility by increasing its expenditure on children beyond the minimum. It chooses to increase its expenditures when there are more children present and when the children are older. The extra total expenditures for all children increases from \$400/year at ages 20–24 to \$1,100/year at ages 45–54. When children are older the family is likely to be engaging in expenditure intensive activities like sending the children to college. At age 21 the children grow up and leave the family unit.

3. The total number of children ever born to the family is 2.8. Since it is not possible for a family to have a fraction of a child, the simulation can be interpreted as an average number that a family with the given parameters would choose. The family desires 0.8 children in the first 5 years, 0.5 children in each of the next two five-year periods and one child in the last period it can have children.

The family maximizes utility by spreading out the birthdates of its children over the child-bearing ages. As the cost of children rises, or preferences for leisure increases, the number of children in the family falls, *ceterius paribus*.

4. The husband and wife were assigned identical parameters in the model with respect to their preferences and production possibilities regarding the generation of leisure, child care and work. The result that the husband spends more time working and less time on child care results exclusively from the differences in initial earnings potential of the husband and wife, and can be viewed as resulting from economic benefits to "trade" between the husband and wife. Changing the parameters to represent a greater ability for child care on the part of the wife would widen the differentials even more.

5. The wife reduces her time in the labor force very little when she has small children. However her participation first falls then rises then falls. If she is assumed

[4] In other simulations as the degree of the C_1 production function was increased toward 1, the incentive to bunch consumption became more pronounced and the path of X_1 started to rise over time.

to be more effective in the production of child care she would specialize and she would spend much less time in the labor force when young children are present.

6. The husband and wife spend only about 25 percent of their time on leisure during the child-bearing years and almost 100 percent of their time on leisure at retirement at age 60. The percentage of time spent on child care increased during the child-bearing years as more children were born but decreased thereafter as the average age of children ages.

7. In the model the initial conditions on potential income of the husband and wife respectively are set at $9,900 and $6,000 per year. Initially the wife earns 60.6 percent of what the husband can earn. Because both the husband and wife work throughout their lifetime, this differential remains approximately constant during their working life. In other simulations where the wife drops out of the labor force earlier, her potential wage falls substantially relative to her husband's.

To illustrate the effect of specialization by the husband in the labor market and the wife on child care or leisure we ran a simulation which resulted in the time paths shown in Table 2 for percent of time worked. We only show the simulation for the period before biological depreciation had a significant effect.

TABLE 2

Age	20–24	25–29	30–34	35–39	40–44	45–49
Work Head (% Time)	49%	47%	46%	45%	32%	26%
Work Wife (% Time)	33%	32%	32%	17%	0%	0%
Income (Husband)[a]	9.9	11.9	14.2	16.7	19.2	20.6
Income (Wife)[a]	6.0	7.2	8.5	10.1	11.5	11.6

[a] Potential income in thousands of dollars/year.

This time path might occur if the family waits till the last possible moment to have children. We would see participation by the wife dropping off sharply in the last possible child bearing ages. It is interesting to contrast the husband-wife income differential with Table 1. The income differential between the husband and wife is about 65 percent at the beginning of marriage. If the wife decreases her participation in the labor force and leaves the labor force for a few years her potential earnings will depreciate while her husband's income is increasing leading to a earning potential ratio of almost 2:1 by age 45. While the parameters were chosen to exaggerate the human capital phenomenon, it offers an explanation for some of the male-female differential.

The time paths shown in Table 2 could be generated by several alternative assumptions. In the actual run X_1 and leisure of the husband and wife were assumed to be related by a Cobb–Douglas production function instead of the CES given by equation (10). In theory setting B_2 to zero in (10) leads to the Cobb–Douglas. However, this creates numerical evaluation problems. In the Table 1, simulation B_2 was 0.10. Other obvious parameter changes which would lead to the widening of the differential have been previously discussed in the text.

8. The family initially borrows in order to consume as evidenced by negative assets until age 44. By age 35–44 assets reach a minimum of –$12,400. Then they start to increase and reach a maximum at age 55–64. After which they again fall to $4,600 at death at age 70.

IV. Summary and Suggestion for Future Study

The present study is suggestive of the role a control model can play in explaining a lifetime household model, explaining time allocation and consumption. The model portrays the family's behavior with the assumed parameters if it chose to maximize consumption throughout its lifetime.

Many of the parameters which were used in the model could not be verified. Future empirical research might yield more precise estimates of some of these parameters. However, the major use of the model is to examine interdependencies in various aspects of life cycle planning, and future work might begin by experimenting with joint changes in various parameters (including initial conditions). Examples of interesting experiments include the effect of lowering initial potential earnings of both parents, increasing the initial relative earnings between the husband and wife, increasing the minimum cost of having children and increasing the interest rate.

Because of the complexity of the model, the quantitative path of any such experiment is very sensitive to the other parameters in the model. For example, any of the above experiments is not likely to affect consumption not related to children very much but will make a big difference in the way time is allocated between work, leisure and child care. On the other hand, the qualitative paths of the model tend to respond very consistently with respect to the above experiments. For example, increasing the cost of children does lead to the obvious result of fewer children born. Increasing the interest rate makes it less profitable for the family to incur negative assets and leads to more work and more terminal assets, and narrowing the wife-husband wage differential tends to equate their time allocations.

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Appendix

The specific parameters used in the Table 1 simulation are presented in this appendix.

For equation (1′) the following parameters were used.

$\gamma_0 = 9{,}000.0$	$\underline{M_t}$
$\beta_1 = 0.90$	$\beta_6 = 9.0$
$\beta_2 = 0.10$	$\beta_6 = 18.0$
$\beta_3 = 10^{10}$	$\beta_7 = 0.9$
$\gamma_1 = 0.25$	$\underline{P_T}$
$\gamma_4 = 0.01$	$\gamma_2 = 98.0$
$\beta_8 = 0.99$	$\beta_4 = 0.5$
$\beta_9 = 0.9$	$\beta_5 = -0.9$
$\gamma_5 = 0.9$	

For equation (4) P_1 and P_2 were set at 1. $P_4(A_t)A_t$ was given by

$$r = 0.5\left[\frac{(0.8)^{(A^*+30)}}{1+(0.8)^{(A^*+30)}}\right] + 0.20$$

where

$$A_t^* = 1_t^h Y_t^h + 1_t^w Y_t^w - X_{1t} - X_{2t} - P_3 Z_t \quad \text{and} \quad A_t - A_{t-1} = A_t^*(1+r)$$

For equations (5) and (6)

$$\alpha_1 = \alpha_4 = -1.50$$

$$\alpha_2 = \alpha_5 = 1.2$$

$$\alpha_3 = \alpha_6 = 0.97$$

$$\delta_2(t) = \delta_3(t) = \frac{-0.97T^4}{10{,}000.0}$$

For equation (1),

$$f(A_T) = 0.25\left[\ln\left(\frac{2.0}{1.0 + 0.8^{(A_T/10.0)}}\right)\right]$$

X_1, X_2 are treated as consumption per 5 year period in the simulation. All dollar amounts were then deflated by 2 to scale dollar amounts to what a typical family would be spending. Dollar amounts are reported in thousands of dollars.

To reflect the appropriate cost of having children in the parameters, N was redefined as twice the number of children born/period. This also implies that the family could not have more than 2.5 children in a five year period without penalty. All of these adjustments have been made for the discussion in the text and tables in the body of the paper.

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